AN ALGEBRAIC CHARACTERIZATION OF MINKOWSKI SPACE

By

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We give an algebraic characterization of three-dimensional and four-dimensional Minkowski space. We construct both spaces from a set of involution elements and the group it generates. We then identify the elements of the original generating set with spacelike lines and their corresponding reflections in the three-dimensional case and with spacelike planes and their corresponding reflections in the four-dimensional case. Further, we explore the relationship between these characterizations and the condition of geometric modular action in algebraic quantum field theory.

CHAPTER 1
INTRODUCTION

The research program which this dissertation is a part of started with a paper by Detlev Buchholz and Stephen J. Summers entitled "An Algebraic Characterization of Vacuum States in Minkowski Space" [9]. In 1975, Bisognano and Wichmann [5] showed that for quantum theories satisfying the Wightman axioms the modular objects associated by Tomita-Takesaki theory to the vacuum state and local algebras generated by field operators with support in wedgelike spacetime regions in Minkowski space have geometrical meaning. Motivated by this work, Buchholz and Summers gave an algebraic characterization of vacuum states on nets of C^*-algebras over Minkowski space and reconstructed the spacetime translations with the help of the modular structures associated with such states. Their result suggested that a "condition of geometric modular action" might hold in quantum field theories on a wider class of spacetime manifolds.

To explain the abstract version of this condition, first some notation is introduced and the basic set-up is given. Let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of C^*-algebras labeled by the elements of some index set I such that $(I,\leq)$ is an ordered set and the property of isotony holds. That is, if $i_1, i_2 \in I$ such that $i_1 \leq i_2$, then $\mathcal{A}_{i_1} \subseteq \mathcal{A}_{i_2}$. Let $\mathcal{A}$ be a C^*-algebra containing $\{\mathcal{A}_i\}_{i\in I}$. It is also required that the assignment $i \mapsto \mathcal{A}_i$ is an order-preserving bijection.

In algebraic quantum field theory, the index set I is usually a collection of open subsets of an appropriate spacetime $(\mathcal{M}, g)$. In such a case, the algebra $\mathcal{A}_i$ is interpreted as the C^*-algebra generated by all the observables measured in a

spacetime region i. Hence, to different spacetime regions should correspond different algebras.

Given a state ω on $\mathcal{A}$, let $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ be the corresponding GNS representation and let $\mathcal{R}_i \equiv \pi_\omega(\mathcal{A}_i)''$, $i \in I$, be the von Neumann algebras generated by the $\pi_\omega(\mathcal{A}_i)$, $i \in I$. Assume that the map $i \mapsto \mathcal{R}_i$ is an order-preserving bijection and that the GNS vector Ω is cyclic and separating for each algebra $\mathcal{R}_i$, $i \in I$. From Tomita-Takesaki theory, we thus have a collection $\{J_i\}_{i \in I}$ of modular involutions and a collection $\{\Delta_i\}_{i \in I}$ of modular operators *directly derivable* from the state and the algebras. Each J_i is an anti-linear involution on $\mathcal{H}_\omega$ such that $J_i \mathcal{R}_i J_i = \mathcal{R}_i{}'$ and $J_i \Omega = \Omega$.

In addition, the set $\{J_i\}_{i \in I}$ generates a group $\mathcal{J}$ which becomes a topological group in the strong operator topology on $\mathcal{B}(\mathcal{H}_\omega)$, the set of all bounded operators on $\mathcal{H}_\omega$. The modular operators $\{\Delta_i\}_{i \in I}$ are positive (unbounded) invertible operators such that

$$\Delta_j^{it} \mathcal{R}_j \Delta_j^{-it} = \mathcal{R}_j, \;\; j \in I, \;\; t \in \mathbb{R}, \;\; i = \sqrt{-1} \;\; \text{and} \;\; \Delta^{it}\Omega = \Omega.$$

In algebraic quantum field theory the state ω models the preparations in the laboratory and the algebras $\mathcal{A}_i$ model the observables in the laboratory and are therefore, viewed as idealizations of operationally determined quantities. Since Tomita-Takesaki theory uniquely gives these modular objects corresponding to $(\mathcal{R}_j, \Omega)$, it thus follows that these modular objects can be viewed as operationally determined.

Motivated by the earlier work of Bisognano and Wichmann [5], Buchholz and Summers [9] proposed that physically interesting states could be selected by looking at those states which satisfied the condition of geometric modular action, CGMA. Given the structures indicated above, the pair $(\{\mathcal{A}_i\}_{i \in I}, \omega)$ satisfies the abstract version of the CGMA if $\{\mathcal{R}_i\}_{i \in I}$ is left invariant under the adjoint action of the

modular conjugations $\{J_i\}_{i\in I}$; that is, if for every i,j in I there is a k in I such that

$$adJ_i(\mathcal{R}_j) \equiv J_i\mathcal{R}_jJ_i = \mathcal{R}_k, \quad \text{where } J_i\mathcal{R}_jJ_i = \{J_iAJ_i : A \in \mathcal{R}_j\}.$$

Thus, for each i in I, there is an order-preserving bijection, automorphism, τ_i on I, $(I,\leq)$, such that $J_i\mathcal{R}_jJ_i = \mathcal{R}_{\tau_i(j)}$, for $j \in I$. The set $\{\tau_i\}_{i\in I}$ is a set of involutions which generate a group $\mathcal{T}$, which is a subgroup of the group of translations on I. Buchholz, Dreyer, Florig, and Summers [6] have shown that the groups $\mathcal{T}$ arising in this manner satisfy certain structure properties, but for the purposes of this thesis, it is only emphasized that $\mathcal{T}$ is generated by involutions and is hence, a Coxeter group.

Thus there are two groups generated by involutions operating on two different levels.

1. The group $\mathcal{T}$ acting on the index set I.
2. The group $\mathcal{J}$ acting on the set $\{\mathcal{R}_i\}_{i\in I}$.

To elaborate further the relation between the groups $\mathcal{T}$ and $\mathcal{J}$, consider the following.

Proposition 1.1 [6] The surjective map $\xi : \mathcal{J} \mapsto \mathcal{T}$ given by $\xi(J_{i_1}\cdots J_{i_m}) = \tau_{i_1}\cdots\tau_{i_m}$, is a group homomorphism. Its kernel $\mathcal{S}$ lies in the center of $\mathcal{J}$ and the adjoint action of $\mathcal{S}$ leaves each $\mathcal{R}_i$ fixed. ■

Thus, $\mathcal{J}$ is a central extension of the group $\mathcal{T}$ by $\mathcal{S}$.

As an immediate consequence of this proposition, $\mathcal{J}$ provides a projective representation of $\mathcal{T}$ with coefficients in an abelian group $\mathcal{Z}$ in the center of $\mathcal{J}$. Thus, the condition of geometric modular action induces a transformation group on the index set I and provides it with a projective representation.

With this in mind, the following program was then posed. Given the operational data available from algebraic quantum field theory, can one determine the spacetime symmetries, the dimension of the spacetime, and the spacetime itself? That is to say, given a net of C^*-algebras and a state ω satisfying the CGMA, can

one determine the spacetime symmetries, the dimension of the spacetime, and even the spacetime itself?

Part of this has been carried out by Buchholz, Dreyer, Florig, and Summers for Minkowski space and de Sitter space [6]. However, in order to do so, they had to presume the respective spacetime as a topological manifold. But would it not be possible to completely derive the spacetime from the operationally given data without any assumption about dimension or topology?

As was pointed out by Dr. Summers, a possibility to do so was opened up in this program in the following manner. As already seen, the CGMA yields an involution generated group complete with a projective representation and there is in the literature a way of deriving spacetimes from such groups going under the name of absolute geometry.

In general, absolute geometry refers to a geometry that includes both Euclidean and non-Euclidean geometry as special cases. Thus, one has a system of axioms not yet implying any decision about parallelism. In our case, the axioms are given in terms of a group of motions as an extension of Klein's Erlangen Program. A group of motions is defined as a set $\mathcal{G}$ of involution elements closed under conjugation and the group $\mathfrak{G}$ it generates. In a group of motions the representations of geometric objects and relations depend only on the given multiplication for the group elements, without reference to any additional structure. The system of axioms is formulated in terms of the involutory generators alone, so that geometric concepts like point, line, and incidence no longer are primary but are derived.

The necessary means for setting up this representation are provided by the totality of reflections in points, lines, and planes (a subset of the set of motions). Points, lines, and planes are in one-to-one correspondence with the reflections in them so that geometric relations among points, lines, and planes correspond to group-theoretic equations among the reflections. This enables one to be able to

formulate geometric theorems about elements of the group of motions and to be able to then prove these theorems by group-theoretic calculation.

To summarize, we are to find conditions on an index set I and a corresponding net of C^*-algebras $\{\mathcal{A}_i\}_{i\in I}$ as well as a state ω satisfying the CGMA such that the elements of I can be naturally identified with open sets of Minkowski space and such that the group $\mathcal{T}$ is implemented by the Poincaré group on this Minkowski space. Out of the group $\mathcal{T}$ we wish to construct Minkowski space such that $\mathcal{T}$'s natural action on Minkowski space is that of the Poincaré group.

This involves two steps. First we carry out the absolute geometry program for three- and four-dimensional Minkowski space. That is, characterize three- and four-dimensional Minkowski space in terms of a group of motions $(\mathcal{G}, \mathfrak{G})$. Second, we must determine what additional structure on the ordered set I would yield from Tomita-Takesaki theory algebraic relations among the J_i (and hence, among the τ_i) which coincide with the algebraic characterization found in step one.

The organization of the thesis is as follows: in Chapter 2, the given pair $(\mathcal{G}, \mathfrak{G})$ is used to construct a three-dimensional Minkowski space out of the plane at infinity. Then identification of the involutory elements of $\mathcal{G}$ with spacelike lines and their group action in $\mathfrak{G}$ with reflections about spacelike lines is made.

In Chapter 3 using the same initial data, $(\mathcal{G}, \mathfrak{G})$, as was given in Chapter 2 but satisfying different axioms, a four-dimensional Minkowski space is constructed. The approach taken here differs from that taken in Chapter 2. This time the affine space is constructed first and then the hyperplane at infinity is used to obtain the metric. The identification of the elements of $\mathcal{G}$ with spacelike planes and their group action in $\mathfrak{G}$ with reflections about spacelike planes is made.

In Chapter 4 a concrete example of the three-dimensional characterization is given. As already mentioned, Bisognano and Wichmann showed that for quantum field theories satisfying the Wightman axioms the modular objects associated by

Tomita-Takesaki theory to the vacuum state and local algebras in wedgelike regions in three-dimensional Minkowski space have geometrical interpretation [5]. In particular, the modular conjugations, $\{J_i\}_{i \in I}$, act as reflections about spacelike lines. In this chapter it is shown that if one chooses the set of wedgelike regions as the index set I, the group $\mathcal{J}$ generated by the set $\{J_i\}_{i \in I}$ satisfies the axiom system given in Chapter 2 for the construction of a three-dimensional Minkowski space.

In Chapter 5 some concluding remarks about the second step described above are made. It is noted that if one assumes the modular stability condition [6] and the half-sided modular inclusion relations given by Wiesbrock [29], then one does obtain a unitary representation of the 2+1-dimensional Poincaré group.

CHAPTER 2
A CONSTRUCTION OF THREE-DIMENSIONAL MINKOWSKI SPACE

In this chapter we give an absolute geometric, that is, an algebraic, characterization of three-dimensional Minkowski space. This chapter is a version of a preprint by the author entitled "A Group-Theoretic Construction Of Minkowski 3-Space Out Of The Plane At Infinity" [28]. Along with the well-known mathematical motivations [1, 2] there are also physical motivations, as we discussed in Chapter 1. Three-dimensional Minkowski space is an affine space whose plane at infinity is a hyperbolic projective-metric plane [12]. In "Absolute Geometry" [2], Bachmann, Pejas, Wolff, and Bauer (BPWB) took an abstract group $\mathfrak{G}$ generated by an invariant system $\mathcal{G}$ of generators in which each of the generators was involutory, satisfying a set of axioms and constructed a hyperbolic projective-metric plane in which the given group $\mathfrak{G}$ was isomorphic to a subgroup of the group of congruent transformations (motions) of the projective-metric plane. By interpreting the elements of $\mathcal{G}$ as line reflections in a hyperbolic plane, BPWB showed that the hyperbolic projective-metric plane could be generated by these line reflections in such a way that these line reflections form a subgroup of the motions group of the projective-metric plane.

Coxeter showed in [13] that every motion of the hyperbolic plane is generated by a suitable product of orthogonal line reflections, where an orthogonal line reflection is defined as a harmonic homology with center exterior point and axis the given ordinary line and where the center and axis are a pole-polar pair. Here we show that Coxeter's and BPWB's notions of motions coincide in the hyperbolic

projective-metric plane and that the motions can be viewed as reflections about exterior points.

Next we embed our projective-metric plane into a three-dimensional projective space. By singling out our original plane as the plane at infinity, we obtain an affine space whose plane at infinity is a hyperbolic projective-metric plane, three-dimensional Minkowski space. Finally, we show that the motions of our original plane induce motions in the affine space and, by a suitable identification, we show that any motion in Minkowski space can be generated by reflections about spacelike lines. Thus, to construct a three-dimensional Minkowski space, one can start with a generating set $\mathcal{G}$ of reflections about spacelike lines in the plane at infinity. So $\mathcal{G}$ may be viewed as a set of reflections about exterior points in a hyperbolic projective-metric plane. Out of the plane at infinity, one can obtain a three-dimensional affine space with the Minkowski metric, which is constructed from a group generated by a set of even isometries or rotations.

The approach in this chapter differs from the method used by Wolff [30] for two-dimensional Minkowski space and by Klotzek and Ottenburg [19] for four-dimensional Minkowski space. The approach in these papers is to begin by constructing the affine space first. For Wolff's [30] two-dimensional case, the elements of the generating set $\mathcal{G}$ are identified with line reflections in an affine plane. For Klotzek and Ottenburg's [19] four-dimensional case, the elements of the generating set $\mathcal{G}$ are identified with reflections about hyperplanes in an affine space. Thus, in each of these papers, the generating set $\mathcal{G}$ is identified with a set of symmetries or odd isometries. A map of affine subspaces is then obtained using the definition of orthogonality given by commuting generators. This map induces a hyperbolic polarity in the hyperplane at infinity, yielding the Minkowski metric.

To briefly recap the two approaches described above, note that both approaches, ours and the one given by Klotzek and Ottenburg [19] and by Wolff

[30], start with a generating set $\mathcal{G}$ of involution elements. In our approach, one can identify the elements of $\mathcal{G}$ with a set of even isometries (rotations) and use the definition of orthogonality induced by the commutation relations of the generators in the hyperplane at infinity to obtain the polarity and then embed this in an affine space to get Minkowski space. In the approach of Klotzek and Ottenburg [19] and Wolff [30], one can identify the elements of $\mathcal{G}$ with a set of odd isometries (symmetries), construct an affine space first, and then use the definition of orthogonality induced by the commutation relations of the generators in the affine space to obtain a polarity in the hyperplane at infinity to get Minkowski space.

2.1 Preliminaries

The starting point for an algebraic characterization of Minkowski space is therefore far from unique. Our particular choice of algebraic characterization, in terms of reflections about spacelike lines in three dimensional Minkowski space, is motivated by physical considerations [6] which we briefly explain in the conclusion.

A hyperbolic projective-metric plane is a projective plane in which a hyperbolic polarity is singled out and used to define orthogonality in the plane. A polarity is an involutory projective correlation. A correlation is a one-to-one mapping of the set of points of the projective plane onto the set of lines, and of the set of lines onto the set of points such that incidence is preserved. A projective correlation is a correlation that transforms the points Y on a line b into the lines y' through the corresponding point B'. So, in general, a correlation maps each point A of the plane into a line a of the plane and maps this line into a new point A'. When the correlation is involutory, A' always coincides with A. Thus a polarity relates A to a, and vice versa. A is called the pole of a and a is called the polar of A. Since this is a projective correlation, the polars of all the points on a form a projectively related pencil of lines through A.

The polarity dualizes incidences: if A lies on b, then the polar of A, a, contains the pole of b, B. In this case we say that A and B are conjugate points, and that a and b are conjugate lines. If A and a are incident, then A and a are said to be self-conjugate: A on its own polar and a through its own pole [14]. A hyperbolic polarity is a polarity which admits self-conjugate points and self-conjugate lines. The set of all self-conjugate points is called a conic, which we shall call the absolute.

In a projective plane in which the theorem of Pappas and the axiom of Fano hold, the polarity can be used to introduce a metric into the plane. Orthogonality is defined as follows: two lines (or two points or a line and a point) are said to be orthogonal or perpendicular to each other if they are conjugate with respect to the polarity.

Congruent transformations of the plane are those collineations of the plane which preserve the absolute; that is, those collineations which leave the absolute invariant. In a projective plane with a hyperbolic polarity as absolute, the group of all collineations in the plane leaving the absolute invariant is called the hyperbolic metric group and the corresponding geometry is called the hyperbolic metric geometry in the plane [31].

The conic or absolute, separates the points of the projective plane into three disjoint classes: ordinary or interior points, points on the absolute, and exterior points. The lines of the projective plane are likewise separated into three disjoint classes. Secant lines, lines that contain interior points, exterior points, and precisely two points on the absolute. Exterior lines, lines that contain only exterior points. And tangent lines, lines that meet the absolute in precisely one point and in which every other point is an exterior point.

Definition 2.1.1. Two lines containing ordinary points, two secant lines, are said to be *parallel* if they have a point of the absolute in common.

Remark. The set of all interior points and the set of lines formed by intersecting

secant lines with the set of interior points, ordinary lines, is classical hyperbolic plane geometry.

2.2 Construction of Π

In this section we list the axioms and main results of BPWB [2] and provide a sketch of some of the arguments they used which are pertinent to this work. For detailed proofs, one is referred to the work of BPWB [2].

Definition 2.2.1. A set of elements of a group is said to be an *invariant system* if it is mapped into itself (and thus onto itself) by every conjugation by an element of the group. An element α of a group $\mathfrak{G}$ is called an *involution* if $\alpha^2 = 1_{\mathfrak{G}}$, where $1_{\mathfrak{G}}$ is the identity element of the group $\mathfrak{G}$.

Basic assumption: A given group $\mathfrak{G}$ is generated by an invariant system $\mathcal{G}$ of involution elements.

The elements of $\mathcal{G}$ are denoted by lowercase Latin letters. Those involutory elements of $\mathfrak{G}$ that can be represented as ab, where $a, b \in \mathcal{G}$, are denoted by uppercase Latin letters. If $\xi, \eta \in \mathfrak{G}$ and $\xi\eta$ is an involution, we denote this by $\xi|\eta$.

Axioms

Axiom 1: *For every P and Q there is a g with $P, Q \,|g$.*

Axiom 2: *If $P, Q \,|g, h$ then $P = Q$ or $g = h$.*

Axiom 3: *If $a, b, c \,|P$ then $abc = d \in \mathcal{G}$.*

Axiom 4: *If $a, b, c \,|g$ then $abc = d \in \mathcal{G}$.*

Axiom 5: *There exist g, h, j such that $g \,|h$ but $j \not| \; g, h, gh$.*

Axiom 6: *There exist elements $d, a, b \in \mathcal{G}$ such that $d, a, b \not| \; P, c$ for $P, c \in \mathfrak{G}$. (There exist lines which have neither a line nor a point in common.)*

Axiom 7: *For each P and for each g there exist at most two elements $h, j \in \mathcal{G}$ such that $P|h, j$ but $g, h \not| \; A, c$ and $g, j \not| \; B, d$ for any $A, B, c, d \in \mathfrak{G}$ (that is, have neither a point nor a line in common).*

Axiom 8: *One never has* $P = g$.

We call the set of axioms just given *axiom system* $\mathcal{A}$, denoted by $as - \mathcal{A}$.

The initial interpretation of the elements of $\mathcal{G}$ is as secant or ordinary lines in a hyperbolic plane for BPWB [2]. In our approach, we view the elements of $\mathcal{G}$ initially as exterior points in a hyperbolic plane. After embedding our hyperbolic projective-metric plane into an affine space, we can identify the elements of $\mathcal{G}$, our generating set, with spacelike lines and their corresponding reflections in a three-dimensional Minkowski space. By realizing that statements about the geometry of the plane at infinity correspond to statements about the geometry of the whole space where all lines and all planes are considered through a point, we see that the axioms also are statements about spacelike lines, the elements of $\mathcal{G}$, and timelike lines, the elements P of $\mathfrak{G}$, through any point in three-dimensional Minkowski space.

The models of the system of axioms are called groups of motions; that is, a group of motions is a pair $(\mathfrak{G}, \mathcal{G})$ consisting of a group $\mathfrak{G}$ and a system $\mathcal{G}$ of generators of the group $\mathfrak{G}$ satisfying the basic assumption and the axioms.

To give a precise form to the geometric language used here to describe group-theoretic concepts occurring in the system of axioms, we associate with the group of motions $(\mathfrak{G}, \mathcal{G})$ the group plane $(\mathfrak{G}, \mathcal{G})$, described as follows.

The elements of $\mathcal{G}$ are called lines of the group plane, and those involutory group elements that can be represented as the product of two elements of $\mathcal{G}$ are called points of the group plane. Two lines g and h of the group plane are said to be perpendicular if $g \mid h$. Thus, the points are those elements of the group that can be written as the product of two perpendicular lines. A point P is incident with a line g in the group plane if $P \mid g$. Two lines are said to be parallel if they satisfy Axiom 6. Thus, if $P \neq Q$, then by Axiom1 and Axiom 2, the points P and Q in the group plane are joined by a unique line. If $P \nmid g$ then Axiom 7 says that there are at most two lines through P parallel to g.

Lemma 2.2.2.[2] *For each* $\alpha \in \mathfrak{G}$, *the mappings* $\sigma_\alpha : g \mapsto g^\alpha \equiv \alpha g \alpha$ *and* $\sigma_\alpha : P \mapsto P^\alpha \equiv \alpha P \alpha$ *are one-to-one mappings of the set of lines and the set of points, each onto itself in the group plane.*

Proof: Let $\alpha \in \mathfrak{G}$, and consider the mapping $\gamma \mapsto \gamma^\alpha \equiv \alpha\gamma\alpha$ of $\mathfrak{G}$ onto itself. It is easily seen that this mapping is bijective. Because $\mathcal{G}$ is an invariant system ($a^b \in \mathcal{G}$ for every $a, b \in \mathcal{G}$) $\mathcal{G}$ is mapped onto itself, and if P is a point, so that $P = gh$ with $g|h$, then $P^\alpha = g^\alpha h^\alpha$ and $g^\alpha | h^\alpha$, so that P^α is also a point. Thus, $g \mapsto g^\alpha$, $P \mapsto P^\alpha$ are one-to-one mappings of the set of lines and the set of points, each onto itself in the group plane. ■

Definition 2.2.3. A one-to-one mapping α of the set of points and the set of lines each onto itself is called an *orthogonal collineation* if it preserves incidence and orthogonality.

Since the "|" relation is preserved under the above mappings, the above mappings preserve incidence and orthogonality as defined above.

Corollary 2.2.4.[2] *The mappings*

$$\sigma_\alpha : g \mapsto g^\alpha \text{ and } \sigma_\alpha : P \mapsto P^\alpha$$

are orthogonal collineations of the group plane and are called motions of the group plane induced by α.

In particular, if α is a line a we have a reflection about the line a in the group plane, and if α is a point A, we have a point reflection about A in the group plane.

If to every $\alpha \in \mathfrak{G}$ one assigns the motion of the group plane induced by α, one obtains a homomorphism of $\mathfrak{G}$ onto the group of motions of the group plane. Bachmann [1] showed that this homomorphism is in fact an isomorphism so that points and lines in the group plane may be identified with their respective reflections. Thus, $\mathfrak{G}$ is seen to be the group of orthogonal collineations of $\mathfrak{G}$ generated by $\mathcal{G}$.

Definition 2.2.5. Planes that are representable as an isomorphic image, with respect to incidence and orthogonality, of the group plane of a group of motions $(\mathfrak{G}, \mathcal{G})$, are called *metric planes.*

BPWB showed how one can embed a metric plane into a projective-metric plane by constructing an ideal plane using pencils of lines [2]. We shall now outline how this is done.

Definition 2.2.6. Three lines are said to *lie in a pencil* if their product is a line; that is, a, b, c lie in a pencil if

$$abc = d \in \mathcal{G}. \qquad (*)$$

Definition 2.2.7. Given two lines a, b with $a \neq b$, the set of lines satisfying $(*)$ is called a pencil of lines and is denoted by $G(ab)$, since it depends only on the product ab.

Note that the relation $(*)$ is symmetric, that is, it is independent of the order in which the three lines are taken, since $cba = (abc)^{-1}$ is a line, the invariance of $\mathcal{G}$ implies that $cab = (abc)^{c}$ is a line and that every motion of the group plane takes triples of lines lying in a pencil into triples in a pencil. The invariance of $\mathcal{G}$ also shows that $(*)$ holds whenever at least two of the three lines coincide.

Using the given axioms, BPWB [2] showed that there are three distinct classes of pencils. If $a, b | V$ then $G(ab) = \{c : c \mid V\}$. In this case, $G(ab)$ is called a *pencil of lines with center* V and is denoted by $G(V)$. If $a, b \mid c$ then $G(ab) = \{d : d \mid c\}$. In this case, $G(ab)$ is called a *pencil of lines with axis* c and is denoted by $G(c)$.

By Axiom 6, there exist lines a, b, c which do not have a common point or a common line. Recall that lines of this type are called parallel. Thus, in this case $G(ab) = \{c : c \parallel a, b \text{ where } a \parallel b\}$, which we denote by P_∞.

Using the above definitions of pencils of lines and the above theorems, BPWP [2] proved that an ideal projective plane, Π, is constructed in the following way. An

ideal point is any pencil of lines $G(ab)$ of the metric plane. The pencils $G(P)$ correspond in a one-to-one way to the points of the metric plane. An ideal line is a certain set of ideal points. There are three types:

1. A proper ideal line $g(a)$, is the set of ideal points that have in common a line a of the metric plane.

2. The set of pencils $G(x)$ with $x \mid P$ for a fixed point P of the metric plane, which we denote by $\tilde{P}$.

3. Each set of ideal points that can be transformed by a halfrotation about a fixed point P of the metric plane into a proper ideal line, which we denote by p_∞.

The polarity is defined by the mappings

$$\begin{aligned} G(C) &\mapsto \tilde{C} \quad \text{and} \quad \tilde{C} \mapsto G(C); \\ P_\infty &\mapsto p_\infty \quad \text{and} \quad p_\infty \mapsto P_\infty; \\ G(c) &\mapsto g(c) \quad \text{and} \quad g(c) \mapsto G(c). \end{aligned}$$

Bachmann [1] showed that the resulting ideal plane is a hyperbolic projective plane in which the theorem of Pappus and the Fano axiom hold; that is, it is a hyperbolic projective-metric plane.

In this model, the ideal points of the form $G(P)$ are the interior points of the hyperbolic projective-metric plane. Thus the points of the metric plane correspond in a one-to-one way with the interior points of the hyperbolic projective-metric plane. The ideal points $G(x)$, for $x \in \mathcal{G}$ are the exterior points of the hyperbolic projective-metric plane.

Theorem 2.2.8. *Each $x \in \mathcal{G}$ corresponds in a one-to-one way with the exterior points of the hyperbolic projective-metric plane.*

Proof: Because each line d of the metric plane is incident with at least three points and a point is of the form ab with $a \mid b$, then each $x \in \mathcal{G}$ is the axis of a pencil. From the uniqueness of perpendiculars each $x \in \mathcal{G}$ corresponds in a one-to-one way with the pencils $G(x)$. Hence, each $x \in \mathcal{G}$ corresponds in a one-to-one way with the exterior points of the hyperbolic projective-metric plane. ■

Thus, the axioms can be viewed as axioms concerning the interior and exterior points of a hyperbolic projective-metric plane. The ideal points of the form $G(ab)$ where $a \parallel b$ are the points on the absolute, that is, the points at infinity in the hyperbolic projective-metric plane.

Now consider the ideal lines. A proper ideal line $g(a)$ is a set of ideal points that have in common a line a of the metric plane.

Theorem 2.2.9. *A proper ideal line* $g(a)$ *is a secant line of the form*

$$g(a) = \{P, x, G(bc) : x, P | a \text{ and } abc \in \mathcal{G} \text{ where } b \parallel c\}.$$

Proof: Every two pencils of lines of the metric plane has at most one line in common. By Axiom 7, each line belongs to at most two pencils of parallels and each line $g \in \mathcal{G}$ belongs to precisely two such pencils. Thus, a proper ideal line contains two points on the absolute, interior points, and exterior points; that is, a proper ideal line is a secant line. If we identify the points P with the pencils $G(P)$ and the lines x with the pencils $G(x)$, then a secant line is the set $g(c) = \{P, x, G(ab) : x, P | c$ and $abc \in G$ where $a \parallel b\}$. ■

Corollary 2.2.10. *The ideal line which consists of pencils* $G(x)$ *with* $x | P$ *for a fixed point* P *of the metric plane consists of only exterior points; that is, it is an exterior line. Under the identification of* x *with* $G(x)$ *then* $\tilde{P} = \{x \in \mathcal{G} : x | P\}$.

The last type of ideal line is a tangent line. It contains only one point, $G(ab) = P_\infty$, on the absolute. Denoting this line by p_∞, then $p_\infty = \{G(ab)\} \cup \{x \in \mathcal{G} : abx \in \mathcal{G}\}$, where $a \parallel b$. Recalling that each $x \in \mathcal{G}$ corresponds to an exterior point in the hyperbolic projective-metric plane, we see that a tangent line consists of one point on the absolute and every other point is an exterior point.

Also note that under the above identifications, each secant line $g(c)$ corresponds to a unique "exterior point", c, $c \notin g(c)$ since one only considers those

$x, P \mid c$ such that $xc \neq 1_{\mathfrak{G}}$ and $Pc \neq 1_{\mathfrak{G}}$. Each exterior line corresponds to a unique interior point P and each tangent line corresponds to a unique point on the absolute.

Theorem 2.2.11. *The map Φ given by*

(i) $\Phi(c) = g(c), \quad \Phi(g(c)) = c$

(ii) $\Phi(P) = \tilde{p}, \quad \Phi(\tilde{p}) = P$

(iii) $\Phi(p_\infty) = P_\infty,\ \Phi(P_\infty) = p_\infty$

is a polarity.

Proof: Let $\mathcal{P}$ be the set of all points of Π and $\mathcal{L}$ the set of all lines of Π. From the remarks above it follows that Φ is a well-defined one-to-one point-to-line mapping of $\mathcal{P}$ onto $\mathcal{L}$ and a well-defined one-to-one line-to-point mapping of $\mathcal{L}$ onto $\mathcal{P}$. Next we show that Φ is a correlation and for this it suffices to show that Φ preserves incidence.

Let $g(c) = \{P, x, G(ab) : x, P|c$ and $abc \in \mathcal{G}$ where $a \parallel b\}$ be a secant line. Let $A, B, d, P_\infty \in g(c)$, where $P_\infty = G(ef) = \{x \in \mathcal{G} : xef \in \mathcal{G}$ and $e \parallel f\}$. Then $A, B, d \mid c$ and $cab \in \mathcal{G}$.

$$\Phi(A) = \tilde{A} = \{x : x \mid A\},\quad \Phi(B) = \tilde{B},\quad \Phi(d) = g(d),\quad \Phi(P_\infty) = p_\infty = \{P_\infty\} \cup \{x : efx \in \mathcal{G}\}$$
$$\Phi(g(c)) = c \in \tilde{A} \cap \tilde{B} \cap g(d) \cap p_\infty \quad \text{and} \quad \Phi(g(c)) \in \Phi(A), \Phi(B), \Phi(d), \Phi(P_\infty).$$

Hence, Φ preserves incidence on a secant line. Now consider an exterior line $\tilde{P} = \{x : x|P\}$ and let $a, b \in \tilde{P}$. Then $a, b|P$ and it follows that $P \in g(a) \cap g(b)$; that is, $\Phi(\tilde{P}) \in \Phi(a) \cap \Phi(b)$ and Φ preserves incidence on an exterior line.

Finally, let $p_\infty = \{G(ab)\} \cup \{x : abx \in \mathcal{G}$ where $a \parallel b\}$ be a tangent line. Clearly, since $\Phi(G(ab)) = p_\infty$, then $P_\infty = G(ab) \in p_\infty$. Now suppose that $d \in p_\infty$. Then $abd \in \mathcal{G}$ and

$$\Phi(d) = g(d) = \{A, x, G(ef) : A, x|d \text{ and } def \in \mathcal{G} \text{ where } e \parallel f\}.$$

Thus, $d \in G(ab) \cap G(ef)$ and $P_\infty \in g(d)$. This implies that $d \in p_\infty$ and $\Phi(p_\infty) \in \Phi(d)$. Hence, Φ preserves incidence and is a correlation.

Note also that from the work above, Φ transforms the points Y on a line b into the lines $\Phi(Y)$ through the point $\Phi(b)$. Thus, Φ is a projective correlation. Since $\Phi^2 = 1_{\mathfrak{G}}$, then Φ is a polarity. Moreover, since $\Phi(p_\infty) = P_\infty$ with $P_\infty \in p_\infty$, then Φ is a hyperbolic polarity. ■

Theorem 2.2.12 *The definition of orthogonality given by the polarity agrees with and is induced by the definition of orthogonality in the group plane.*

Proof: If we define perpendicularity with respect to our polarity then the following are true (we use the notation $\leftrightarrow$ to denote the phrase "if and only if"):

(i) $g(c) \perp g(a) \leftrightarrow \Phi(g(c)) = c \in g(a)$ and $\Phi(g(a)) = a \in g(c) \leftrightarrow a \,|c$.

(ii) $g(c) \perp \tilde{P} \leftrightarrow \Phi(g(c)) = c \in \tilde{P} \leftrightarrow c \,|P$.

(iii) $C \perp \tilde{P} \leftrightarrow \Phi(\tilde{P}) = P \in g(c) = \Phi(c) \leftrightarrow P \,|c$.

(iv) $C \perp g(x) \leftrightarrow \Phi(g(x)) = x \in g(c) = \Phi(c) \leftrightarrow x \,|c$. ■

Instead of interpreting our original generators as ordinary lines in a hyperbolic plane, we now interpret them as exterior points. We can construct a hyperbolic projective-metric plane in which the theorem of Pappus and Fano's axiom hold, which is generated by the exterior points of the hyperbolic projective-metric plane.

With the identifications above and the geometric objects above, we show in the next section that the motions of the hyperbolic projective-metric plane above can be generated by reflections about exterior points; that is, any transformation in the hyperbolic plane which leaves the absolute invariant can be generated by a suitable product of reflections about exterior points.

2.3 Reflections About Exterior Points

Definition 2.3.1. A *collineation* is a one-to-one map of the set of points onto the set of points and a one-to-one map of the set of lines onto the set of lines that preserves the incidence relation.

Definition 2.3.2. A *perspective collineation* is a collineation which leaves a line

pointwise fixed, its axis, and a point line-wise fixed, its center.

Definition 2.3.3. A *homology* is a perspective collineation with center a point B and axis a line b where B is not incident with b.

Definition 2.3.4. A *harmonic homology* with center B and axis b, where B is not incident with b, is a homology which relates each point A in the plane to its harmonic conjugate with respect to the two points B and $(b,[A,B])$, where $[A,B]$ is the line joining A and B and $(b,[A,B])$ is the point of intersection of b and $[A,B]$.

Definition 2.3.5. A *complete quadrangle* is a figure consisting of four points (the vertices), no three of which are collinear, and of the six lines joining pairs of these points. If l is one of these lines, called a side, then it lies on two of the vertices, and the line joining the other two vertices is called the opposite side to l. The intersection of two opposite sides is called a diagonal point.

Definition 2.3.6. A point D is the *harmonic conjugate* of a point C with respect to points A and B if A and B are two vertices of a complete quadrangle, C is the diagonal point on the line joining A and B, and D is the point where the line joining the other two diagonal points cuts $[A,B]$. One denotes this relationship by $H(AB,CD)$.

Example 2.3.7. Let $A,B,$ and C be three collinear points. For a quick construction of the harmonic conjugate D of C with respect to A and B let Q,R,S be any points such that $[Q,R],[Q,S],$ and $[R,S]$ pass through A,B,C respectively. Let $\{P\} = [A,S] \cap [B,R]$, then $\{D\} = [A,B] \cap [P,Q]$ ([11]). Note that if $[R,S] \parallel [A,C]$ then D is the midpoint of A and B.

Coxeter [13] showed that any congruent transformation of the hyperbolic plane is a collineation which preserves the absolute and that any such transformation is a product of reflections about ordinary lines in the hyperbolic plane where a line reflection about a line m is a harmonic homology with center M and axis m, where M and m are a pole-polar pair and M is an exterior point. A point

reflection is defined similarly, a harmonic homology with center M and axis m, where M and m are a pole-polar pair, M is an interior point, and m is an exterior line. Note that in both cases, M and m are nonincident.

In keeping with the notation employed at the end of §2.2, let b be an exterior point and $g(b)$ its pole.

Lemma 2.3.8. *The map* $\Psi_b : \left\{ \begin{array}{lllllll} A & \mapsto & A^b & \textit{and} & d & \mapsto & d^b \\ \tilde{A} & \mapsto & \tilde{A}^b & & d & \mapsto & g(d)^b \\ P_\infty & \mapsto & P_\infty^b & & p_\infty & \mapsto & p_\infty^b \end{array} \right\}$ *is a collineation.*

Proof: This follows from the earlier observation that the motions of the group plane map pencils onto pencils preserving the "|" relation. ■

Lemma 2.3.9. Ψ_b *is a perspective collineation and, hence, a homology.*

Proof: Recall that $g(b) = \{A, x, P_\infty : x, A \mid b$ and where b lies in the pencil $P_\infty\}$. For any A and x in $g(b)$ we have $A^b = A$ and $x^b = x$ since $A, x \mid b$ and if $A', x' \notin g(b)$ then $A', x' \nmid b$ and $A^b \neq A$, $x^b \neq x$, and $A^b, x^b \nmid b$. Thus, $A^b, x^b \notin g(b)$.

Recall also that $P_\infty = G(cd)$, where c and d do not have a common perpendicular nor a common point and thus, $G(cd) = \{f : fcd \in \mathcal{G}\}$. Now $g(b)$ is a secant line, so that it contains two such distinct points, P_∞ and Q_∞, say, on the absolute. Since the motions of the group plane map pencils onto pencils preserving the "|" relation it follows that if $c, d \in P_\infty$ then $c^b, d^b \in P_\infty$ and hence, $P_\infty^b = P_\infty$ and $Q_\infty^b = Q_\infty$. Moreover, if $R_\infty \notin g(b)$ then it follows that $R_\infty^b \notin g(b)$. Thus, Ψ_b leaves $g(b)$ pointwise invariant.

Now let $g(d)$, $\tilde{Q}$, and r_∞ be a secant line, exterior line, and tangent line, respectively, containing b. For $e \in g(d)$ we have $e \mid d$ and $e^b \mid d^b = d$ since $b \mid d$, thus $e^b \in g(d)$. For $A \in g(d)$, $A^b \mid d^b = d$, so $A^b \in g(d)$. Similarly, it follows that if $P_\infty \in g(d)$ then $P_\infty^b \in g(d)$, and that $g(d)^b = g(d)$. One easily sees that $\tilde{Q}^b = \tilde{Q}$ and $r_\infty^b = r_\infty$. Thus, Ψ_b leaves every line through b invariant and Ψ_b is a perspective

collineation for each $b \in \mathcal{G}$. ■

Theorem 2.3.10. Ψ_b *is a harmonic homology.*

Proof: Since A^b is again a point in the original group plane and since d^b is again a line in the original group plane and from the observations above, we have, for each $b \in \mathcal{G}$, Ψ_b maps interior points to interior points, exterior points to exterior points, points on the absolute to points on the absolute, secant lines to secant lines, exterior lines to exterior lines, and tangent lines to tangent lines. Moreover, since $(\xi^b)^b = \xi$ for any $\xi \in \mathfrak{G}$, Ψ_b is involutory for each $b \in \mathcal{G}$. Now in a projective plane in which the theorem of Pappus holds, the only collineations which are involutory are harmonic homologies [10], thus Ψ_b is a harmonic homology for each $b \in \mathcal{G}$. ■

Theorem 2.3.11. *Interior point reflections are generated by exterior point reflections.*

Proof: A similar argument shows that for each interior point A, Ψ_A is a harmonic homology with center A and axis $\tilde{A}$ where $\tilde{A}$ is the polar of A, $A \notin \tilde{A}$, and where Ψ_A is defined analogously to Ψ_b. Thus, each Ψ_A is a point reflection and since A is the product of two exterior points, we see that point reflections about interior points are generated by reflections about exterior points. ■

Theorem 2.3.12. *The reflection of an interior point about a secant line is the same as reflecting the interior point about an exterior point. Moreover, since any motion of the hyperbolic plane is a product of line reflections about secant lines, any motion of the hyperbolic plane is generated by reflections about exterior points.*

Proof: Consider a line reflection in the hyperbolic plane; that is, the harmonic homology with axis $g(b)$ and center b. Let A be an interior point and $g(d)$ a line through A meeting b. Since $b \in g(d)$ then $b \mid d$ and $g(d)$ is orthogonal to $g(b)$. Let E be the point where $g(b)$ meets $g(d)$. Since $E \in g(b)$ then $E \mid b$ and $Eb = f$ for some $f \in \mathcal{G}$. It follows that the reflection of A about $g(b)$ is the same as the reflection of A about E. Since $b \mid d$ and $E \mid d$ then $bd = C$ and we have $E, C \mid b, d$ with $b \neq d$. Thus, by Axiom 2, $E = C = bd$. Hence, $A^E = A^{db} = A^b$. ■

Theorem **2.3.13** *Reflections of exterior points about exterior points and about exterior lines are also motions of the projective-metric plane; that is, the* Ψ_b*'s for* $b \in \mathcal{G}$ *acting on exterior points and exterior lines are motions of the hyperbolic projective-metric plane.*

Proof: The motions of the projective-metric plane are precisely those collineations which leave the absolute invariant. ∎

We also point out that the proof that each Ψ_b is an involutory homology also showed that the Fano axiom holds, since in a projective plane in which the Fano axiom does not hold no homology can be an involution [4].

2.4 Embedding a Hyperbolic Projective-Metric Plane

In this section we embed our hyperbolic projective-metric plane into a three dimensional projective space, finally obtaining an affine space whose plane at infinity is isomorphic to our original projective-metric plane. Any projective plane Π in which the theorem of Pappus holds can be represented as the projective coordinate plane over a field $\mathcal{K}$. (The theorem of Pappus guarantees the commutivity of $\mathcal{K}$.) Then by means of considering quadruples of elements of $\mathcal{K}$, one can define a projective space $\mathbb{P}_3(\mathcal{K})$ in which the coordinate plane corresponding to Π is included. If the Fano axiom holds, then the corresponding coordinate field $\mathcal{K}$ is not of characteristic 2 [4]. By singling out the coordinate plane corresponding to Π as the plane at infinity, one obtains an affine space whose plane at infinity is a hyperbolic projective-metric plane: that is, three-dimensional Minkowski space.

To say that a plane Π is a projective coordinate plane over a field $\mathcal{K}$ means that each point of Π is a triple of numbers (x_0,x_1,x_2), not all $x_i = 0$, together with all multiples $(\lambda x_0, \lambda x_1, \lambda x_2)$, for $\lambda \neq 0$ and $\lambda \in \mathcal{K}$. Similarly, each line of Π is a triple of numbers $[u_0,u_1,u_2]$, not all $u_i = 0$, together with all multiples $[\lambda u_0, \lambda u_1, \lambda u_2]$, $\lambda \neq 0$. In $\mathbb{P}_3(\mathcal{K})$, all the quadruples of numbers with the last entry zero correspond

to Π. One can now obtain an affine space $\mathcal{A}$ by defining the points of $\mathcal{A}$ to be those of $\mathbb{P}_3(\mathcal{K}) - \Pi$; that is, those points whose last entry is nonzero; a line l of $\mathcal{A}$ to be a line l' in $\mathbb{P}_3(\mathcal{K}) - \Pi$ minus the intersection point of the line l' with Π; and by defining a point P in $\mathcal{A}$ to be incident with a line l of $\mathcal{A}$ if, and only if, P is incident with the corresponding l'. Planes of $\mathcal{A}$ are obtained in a similar way [14].

Thus, each point in Π represents the set of all lines in $\mathcal{A}$ parallel to a given line, where lines and planes are said to be parallel if their first three coordinates are the same, and each line in Π represents the set of all planes parallel to a given plane. Because parallel objects can be considered to intersect at infinity, we call Π the plane at infinity.

2.5 Exterior Point Reflections Generate Motions in an Affine Space

In this section we state and prove the main result of this chapter. Coxeter showed that three-dimensional Minkowski space is an affine space whose plane at infinity is a hyperbolic projective-metric plane [11]. He also classified the lines and planes of the affine space according to their sections by the plane at infinity as follows:

Line or Plane	Section at Infinity
Timelike line	Interior point
Lightlike line	Point on the absolute
Spacelike line	Exterior point
Characteristic plane	Tangent line
Minkowski plane	Secant line
Spacelike plane	Exterior line

He also showed that if one starts with an affine space and introduces a hyperbolic polarity in the plane at infinity of the affine space, then the polarity induces a Minkowskian metric on the whole space. Under a hyperbolic polarity, a line and a plane or a plane and a plane are perpendicular if their elements at infinity

correspond. Two lines are said to be perpendicular if they intersect and their elements at infinity correspond under the polarity.

Theorem 2.5.1. *Every motion in a three-dimensional affine space with a hyperbolic polarity defined on its plane at infinity, is generated by reflections about exterior points. Moreover, because exterior points correspond to spacelike lines, then any motion in three-dimensional Minkowski space is generated by reflections about spacelike lines.*

Proof: Because any motion in three-dimensional Minkowski space can be generated by a suitable product of plane reflections, it suffices to show that reflections about exterior points generate plane reflections.

Let α be any Minkowski plane or spacelike plane. Let P be any point in Minkowski space. Let l be the line through P parallel to α. Let α_∞ denote the section of α at infinity. Applying the polarity to α_∞ we get a point $g_\infty \perp \alpha_\infty$. Let g be a line through P whose section at infinity is g_∞, so that g is a line through P orthogonal to α. because each line in the plane at infinity contains at least 3 points, there exists a line l in α which is orthogonal to g as $g_\infty \perp \alpha_\infty$. Now let m be a line through P not in α which intersects l. It follows that the reflection of P about α is the same as reflecting m about l and taking the intersection of the image of m under the reflection with g. By the construction of the affine space and the definition of orthogonality in the affine space it follows that l and m must act as their sections at infinity act. because any point reflection in the hyperbolic projective-metric plane can be generated by reflections about exterior points, we have that the reflection of P about α is generated by reflections of P about spacelike lines. ■

2.6 Conclusion

As already indicated above, the geometric model for the generators of $\mathcal{G}$ which lies behind the choice of algebraic characterization of three-dimensional Minkowski

space differs significantly from those of previous absolute geometric characterizations of Minkowski space. The model given here is the set of reflections about spacelike lines, which is not a choice which would be made *a priori* by other mathematicians. However, this is yet another example of a situation where the initial data are imposed by physical, as opposed to purely mathematical, considerations.

In the next chapter starting with the same initial data, but satisfying different axioms, a construction of four-dimensional Minkowski space is given. First the affine space is constructed and then the hyperplane at infinity is used. Also given is an explicit construction of the field, the vector space, and the metric.

CHAPTER 3
A CONSTRUCTION OF FOUR-DIMENSIONAL MINKOWSKI SPACE

In this chapter a construction of four-dimensional Minkowski space will be given using the same initial data as in Chapter 2 but satisfying different axioms. The actual construction is quite long, so to aid the reader in following, A brief outline of the procedure shall be given here. First some general theorems and the basic definitions will be given in the first two sections. In Sections 3.3 and 3.4 attention is restricted to two dimensions in order to obtain the necessary machinery to construct the field. Once the field has been obtained, then a vector space is constructed and given a definition of orthogonality. It is then shown that the vector definition of orthogonality is induced by and agrees with the initial definition of perpendicularity for the geometry generated by the original set of involutions $\mathcal{G}$.

To obtain a metric vector space from the constructed vector space $\mathcal{V}$, a map π is defined on the subspaces of $\mathcal{V}$. The map π is defined to send a subspace to its orthogonal complement. Using the work of [3] (which is given for the convenience of the reader) the Minkowski metric is obtained and hence, Minkowski space. The last section of this chapter identifies the elements of $\mathcal{G}$ with spacelike planes in Minkowski space and the motions of the elements of $\mathcal{G}$ with reflections about spacelike planes, as was required in Chapter 1.

3.1 Preliminaries and General Theorems

Let there be given a nonempty set $\mathcal{G}$ of involution elements and the group $\mathfrak{G}$ it generates, where for any $\alpha \in \mathcal{G}$ and for any $\xi \in \mathfrak{G}$ we have $\alpha^{\xi} \equiv \xi^{-1}\alpha\xi \in \mathcal{G}$. If the product of two distinct elements $\xi_1, \xi_2 \in \mathfrak{G}$ is an involution then we denote this by

writing $\xi_1 | \xi_2$. We note that if $\xi_1 | \xi_2$ then $\xi_1^{\xi} | \xi_2^{\xi}$ for all ξ, ξ_1, ξ_2 in $\mathfrak{G}$ because

$$\xi_1^{\xi}\xi_2^{\xi} = \xi^{-1}\xi_1\xi\xi^{-1}\xi_2\xi = \xi^{-1}\xi_1\xi_2\xi = \xi^{-1}\xi_2\xi_1\xi = \xi_2^{\xi}\xi_1^{\xi}.$$

Let $\mathcal{M} = \{\alpha\beta\colon \alpha \mid \beta \text{ and } \alpha,\beta \in \mathcal{G}\}$ and $\mathcal{P} = \mathcal{G} \cup \mathcal{M}$. We consider the elements of $\mathcal{G}$ as spacelike planes and the elements of $\mathcal{M}$ as Minkowskian or Lorentzian planes. Thus, $\mathcal{P}$ consists of the totality of "non-singular" planes and we denote the elements of $\mathcal{P}$ by $\alpha, \beta, \gamma, \ldots$.

We begin by giving the basic assumption and by making some preliminary definitions. All the axioms are then listed for the convenience of the reader. The geometric meaning of the axioms and the symbols used will be made clear in the appropriate sections. The first four sections examine the incidence axioms. The order axioms are reintroduced in Section 3.5, where we construct and order the field to obtain a field isomorphic to the reals. In Section 3.6 we give the motivation behind the particular choice of dilation axioms. Using these axioms we are able to define a scalar multiplication and thereby obtain an affine vector space. The polarity axioms are given again and examined in Section 3.8, where we define orthogonal vectors.

Basic assumption: *If* $\alpha \in \mathcal{G}$ *and* $\beta \in \mathcal{M}$ *then* $\alpha^{\xi} \in \mathcal{G}$ *and* $\beta^{\xi} \in \mathcal{M}$ *for every* ξ *in* $\mathcal{G}$.

For the following, let $\alpha, \beta \in \mathcal{P}$ with $\alpha | \beta$.

Definition 3.1. If $\alpha, \beta \in \mathcal{G}$, then $\alpha\beta \in \mathcal{M}$ by definition and we write $\alpha \perp \beta$ and we say α *is perpendicular to or orthogonal to* β.

Definition 3.2. Suppose that $\alpha \in \mathcal{G}$, $\beta \in \mathcal{M}$.

(i) If $\alpha\beta \in \mathcal{G}$, then we write $\alpha \perp \beta$ and we say α *is perpendicular to* β.

(ii) If $\alpha\beta \notin \mathcal{G}$, then we write $\alpha \hat{\perp} \beta$ and we say α *is absolutely perpendicular to* β.

Definition 3.3. Let $\mathfrak{X} = \{\alpha\beta : \alpha \hat{\perp} \beta\}$. We call the elements of $\mathfrak{X}$ *points* and we denote these elements by $A, B, C, \ldots$.

Definition 3.4. If $\alpha, \beta \in \mathcal{M}$ and $\alpha\beta \in \mathcal{M}$, then we write $\alpha \perp \beta$ and we say α *is perpendicular to* β.

Definition 3.5. The point A and the plane α are called *incident* when $A|\alpha$. For each $\alpha \in \mathcal{P}$, set $\mathfrak{X}_\alpha \equiv \{A : A|\alpha\}$.se that $A, B|\alpha_1, \alpha_2$ where $A \neq B$ and $\alpha_1 \neq \alpha_2$.

Suppose that $A, B|\alpha_1, \alpha_2$; where $A \neq B$ and $\alpha_1 \neq \alpha_2$. We define the line g containing A and B as

$$g = g_{AB} \equiv [\alpha_1, \alpha_2] = \{C \in \mathfrak{X} : C|\alpha_1, \alpha_2\}.$$

We say that g is the *intersection* of α_1 and α_2 ($\mathfrak{X}_{\alpha_1}$ and $\mathfrak{X}_{\alpha_2}$), $g \subset \alpha_1, \alpha_2$ ($g \subset \mathfrak{X}_{\alpha_1}, \mathfrak{X}_{\alpha_2}$). The point C is *incident* with g, $C \in g$, if $C|\alpha_1, \alpha_2$.

If $A \neq B$ are two points such that there exist α and β with $A, B|\alpha, \beta$ and $\alpha \perp \beta$ then we say that A and B are *joinable* and we write $A, B \in g_{AB} \equiv [\alpha, \beta, \alpha\beta]$. If g is a line which can be put into the form $g = [\alpha, \beta, \alpha\beta]$ where $\alpha \perp \beta$ then we say that g is *nonisotropic.*

If A and B are two distinct points such that there do not exist α, β with $A, B|\alpha, \beta$ and $\alpha \perp \beta$ then we say that A and B are *unjoinable.* If g is a line which cannot be put into the form $g = [\alpha, \beta, \alpha\beta]$ with $\alpha \perp \beta$ then we say that g is *isotropic or null.*

Incidence Axioms

Axiom 1. *For each* P, α *there exists a unique* $\beta \in \mathcal{P}$ *such that* $P|\beta$ *and* $\alpha\beta = Q$.

Axiom 2. *If* $A, B\ |\alpha, \beta, \varepsilon$ *and* $C|\alpha, \beta$ *then* $C|\varepsilon$.

Axiom 3. *If* $P, Q|\alpha, \beta$ *and* $\alpha \hat{\perp} \beta$ *then* $P = Q$.

Axiom 4. *If* $\alpha, \beta, \gamma \in \mathcal{G}$ *are distinct and* $\alpha \perp \beta \perp \gamma \perp \alpha$ *then* $\alpha\beta\gamma \notin \mathcal{G}$.

Axiom 5. *If* $\alpha, \beta \in \mathcal{M}$ *and* $\alpha|\beta$ *then* $\alpha\beta \in \mathcal{M}$.

Axiom 6. *For all* A, B; $A \neq B$, *there exists* α, β *such that* $A, B|\alpha, \beta$ *and* $\alpha \neq \beta$.

Axiom 7. *If* $\alpha \perp \beta$ *then there exists* $A, B|\alpha, \beta$ *such that* $A \neq B$.

Axiom 8. *For all* A, B, C; $ABC = D \in \mathfrak{X}$.

Axiom 9. *If* $O|\alpha, \beta, \gamma, \delta$ *with* $\beta, \gamma, \delta \perp \alpha$ *then* $\beta\gamma\delta = \varepsilon$.

Axiom 10. *If A,B,C are pairwise unjoinable points and $A,B|\alpha$ then $C|\alpha$.*

Axiom 11. *For all $\alpha \in \mathcal{G}$, there exist distinct $\beta,\gamma,\delta \in \mathcal{G}$ such that $\alpha \perp \beta \perp \gamma \perp \alpha$ but $\delta \not\perp \alpha,\beta,\gamma$.*

Axiom 12. *If $A,B|\alpha$; $\alpha \in \mathcal{G}$, then there exists $\beta \in G$ such that $A,B|\beta$ and $\beta \perp \alpha$.*

Axiom 13. *For each $P|\alpha$, $\alpha \in \mathcal{M}$, there are distinct points $A,B|\alpha$ such that $P \neq A,B$ and P is unjoinable with both A and B, but A and B are joinable, and if $C|\alpha$ is unjoinable with P then C is unjoinable with A or with B or $C = A$ or $C = B$.*

Axiom 14. *If A and B are joinable and $A,B|\alpha$, then there exists $\beta \perp \alpha$ such that $A,B|\beta$.*

Axiom 15. *If α,β,γ are distinct with $\beta,\gamma \perp \alpha$ and $A,B|\alpha,\beta,\gamma$; $A \neq B$, then $\alpha = \beta\gamma$ and if $A,B|\gamma$; $\gamma \perp \alpha$ then $\alpha = \beta\gamma$ or $\beta = \gamma$.*

Order Axioms

Axiom F. (Formally Real Axiom)[21] *Let $O,E \mid \alpha$, $\alpha \in \mathcal{M}$ with O and E unjoinable. Let $\lambda,\delta,\eta \in \mathcal{P}$. If $O \mid \lambda,\delta,\eta$ and $\lambda,\delta,\eta \perp \alpha$ then there is a $\gamma \in \mathcal{P}$ such that*

$$O|\gamma, \quad \gamma \perp \alpha, \quad \text{and } E^{\lambda\delta\lambda\delta}OE^{\lambda\eta\lambda\eta} = E^{\lambda\gamma\lambda\gamma}.$$

Axiom L. (LUB) *If $\mathcal{A} \subset \mathcal{K}$, $\mathcal{A} \neq \varnothing$, and $\mathcal{A}$ is bounded above, then there exists an A in $\mathcal{K}$ such that $A \geq X$, for all X in $\mathcal{A}$ and if $B \geq X$ for all X in $\mathcal{A}$ then $A \leq B$.*

Dilation Axioms

Axiom T. *If $O \in t,g$, with $t \neq g$, where t is timelike or t and g are both isotropic, then there exists a unique $\alpha \in \mathcal{M}$* such that $g,t \subset \mathfrak{X}_\alpha$.

Axiom D. *(Desargues) Let g,h,k be any three distinct lines, not necessarily coplanar, which intersect in a point O. Let $P,Q \in g$; $R,S \in h$; and $T,U \in k$. If $g_{PT} \parallel g_{QU}$ and $g_{RT} \parallel g_{SU}$ then $g_{PR} \parallel g_{QS}$.*

Axiom R. *Let $O \in g,h$; $P,Q \in g$, and $R,S \in h$. If $g_{PR} \parallel g_{QS}$ then $g_{O,POR} \parallel g_{O,QOS}$.*

Polarity Axioms

Axiom U. ($U^{\perp}$ subspace axiom) *Let O, A, B, T and C be four points with $O, C \mid \gamma\delta$; $A, O \mid \alpha$; $O, B \mid \beta$; with $\alpha \hat{\perp} \gamma$ and $\beta \hat{\perp} \delta$. Then there exist $\lambda, \varepsilon \in \mathcal{P}$ such that $\lambda \hat{\perp} \varepsilon$; $O, AOB \mid \lambda$; and $O, C \mid \varepsilon$.*

Axiom S1. *If $g \subset \mathfrak{X}_\alpha$, $\alpha \in \mathcal{G}$, $h \subset \mathfrak{X}_\beta$, $\beta \in \mathcal{G}$, and there exist $\gamma, \delta \in \mathcal{P}$ such that $\lambda \hat{\perp} \delta$; $\gamma\delta \in g \cap h$, $g \subset \mathfrak{X}_\gamma$, and $h \subset \mathfrak{X}_\delta$ then there exists $\varepsilon \in \mathcal{G}$ such that $g, h \subset \mathfrak{X}_\varepsilon$. (If g and h are two orthogonal spacelike lines then there is a spacelike plane containing them.)*

Axiom S2. *Let g and h be two distinct lines such that $P \in g \cap h$ but there does not exist $\beta \in \mathcal{P}$ such that $g \subset \mathfrak{X}_\beta$ and $h \subset \mathfrak{X}_\beta$. Then either there are $\alpha, \beta' \in \mathcal{P}$ such that $P = \alpha\beta'$, $g \subset \mathfrak{X}_\alpha$, and $h \subset \mathfrak{X}_{\beta'}$ or for all $A \in g$ there exists $B \in h$ such that P and APB are unjoinable.*

This concludes the list of axioms. Note that by Axiom 5, $\alpha\beta \notin \mathcal{M}$ for $\alpha \hat{\perp} \beta$, $\alpha, \beta \in \mathcal{P}$ and if $\alpha, \beta \in \mathcal{M}$ are distinct with $\alpha|\beta$, then. $\alpha\beta \in \mathcal{M}$. By Axiom 6, every two points is contained in a line.

Notation. Due to the brevity of the theorems and proofs in sections 3.1 through 3.4, we shall follow the usual convention in absolute geometry [1, 2, 19, 30] of simply numbering the results in these sections.

3.1.1. Properties of $\mathcal{M}$.

3.1.1.1. *The set $\mathcal{M} \neq \varnothing$.*

Proof: By assumption $\mathcal{G} \neq \varnothing$, so let $\alpha \in \mathcal{G}$. By Axiom 11, there exists $\gamma \in \mathcal{G}$ such that $\alpha \perp \gamma$. Hence, $\alpha\gamma \in \mathcal{M}$ by definition and $\mathcal{M} \neq \varnothing$. ■

3.1.1.2. *The elements of $\mathcal{M}$ are involutions.*

Proof: Let $\gamma \in \mathcal{M}$. Then we may write $\gamma = \alpha\beta$ where $\alpha, \beta \in \mathcal{G}$ and $\alpha \perp \beta$. We have $\gamma\gamma = \alpha\beta\alpha\beta = \alpha\alpha\beta\beta = 1_{\mathfrak{G}}$. ■

3.1.1.3. *For every $\beta \in \mathcal{M}$ and for every $\xi \in \mathfrak{G}$, $\beta^\xi \in \mathcal{M}$.*

Proof: Let $\beta = \alpha_1\alpha_2$ with $\alpha_1, \alpha_2 \in \mathcal{G}$ and $\alpha_1 | \alpha_2$. By our assumptions on $\mathcal{G}$ and because $\alpha_1 | \alpha_2$ we have for all $\xi \in \mathfrak{G}$, $\alpha_1^{\xi}\alpha_2^{\xi} \in \mathcal{G}$ and $\alpha_1^{\xi} | \alpha_2^{\xi}$ so that $\alpha_1^{\xi}\alpha_2^{\xi} \in \mathcal{M}$. ■

3.1.1.4. *The sets* $\mathcal{G}$ *and* $\mathcal{M}$ *are disjoint,* $\mathcal{G} \cap \mathcal{M} = \varnothing$.

Proof: Let $\mathcal{G}' = \mathcal{G} \backslash \mathcal{M}$. Then $\mathcal{G}'$ consists of involution elements, $\mathcal{G}' \cap \mathcal{M} = \varnothing$, and $\mathcal{P} = \mathcal{G}' \cup \mathcal{M} = \mathcal{G} \cup \mathcal{M}$. Let $\gamma \in \mathcal{G}'$ and $\xi \in \mathfrak{G}$. Now if $\gamma^{\xi} \in \mathcal{M}$, then by 3.1.1.3 above $\gamma = (\gamma^{\xi})^{\xi^{-1}} \in \mathcal{M}$, a contradiction. Thus, $\mathcal{G}'$ is an invariant system of generators and without loss of generality, we may assume that $\mathcal{G} \cap \mathcal{M} = \varnothing$. ■

3.1.1.5. *If* $\alpha \in \mathcal{G}$, $\beta \in \mathcal{M}$ *and* $\alpha \hat{\perp} \beta$, *then* $\alpha\beta \notin \mathcal{M}$.

Proof: If $\alpha\beta = \gamma \in \mathcal{M}$ then $\alpha = \beta\gamma \in \mathcal{G}$ where $\beta, \gamma \in \mathcal{M}$ and $\beta | \gamma$, which contradicts Axiom 5. ■

3.1.2. Properties of $\mathcal{P}$.

For the remainder of this dissertation, let the symbol " $\leftrightarrow$" denote the phrase "if, and only if'.

3.1.2.1. *If* $\alpha, \beta \in \mathcal{P}$ *and* $\xi \in \mathfrak{G}$ *then* $\alpha \perp \beta$ if, and only if, $\alpha^{\xi} \perp \beta^{\xi}$.

Proof: First suppose that $\alpha, \beta \in \mathcal{G}$. Then $\alpha \perp \beta \leftrightarrow \alpha^{\xi} \perp \beta^{\xi}$ because $\alpha \perp \beta$ implies that $\alpha | \beta$ and $\alpha | \beta \leftrightarrow \alpha^{\xi} | \beta^{\xi}$. If $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{M}$ and $\alpha \perp \beta$ then $\alpha\beta = \gamma \in \mathcal{G}$; $\alpha^{\xi}, \gamma^{\xi} \in \mathcal{G}$, $\beta^{\xi} \in \mathcal{M}$, and $\alpha^{\xi}\beta^{\xi} = \gamma^{\xi} \in \mathcal{G}$ implies that $\alpha^{\xi} \perp \beta^{\xi}$. Conversely, if $\alpha^{\xi} \perp \beta^{\xi}$ then $\alpha^{\xi}\beta^{\xi} = \delta \in \mathcal{G}$ and $\alpha\beta = \delta^{\xi^{-1}} \in \mathcal{G}$, so that $\alpha \perp \beta$. Finally, suppose that $\alpha, \beta \in \mathcal{M}$. Then $\alpha^{\xi}, \beta^{\xi} \in \mathcal{M}$ and $\alpha\beta \in \mathcal{M} \leftrightarrow \alpha^{\xi}\beta^{\xi} \in \mathcal{M}$. ■

3.1.2.2. *If* $\alpha, \beta \in \mathcal{P}$ *and* $\xi \in \mathfrak{G}$, *then* $\alpha \hat{\perp} \beta \leftrightarrow \alpha^{\xi} \hat{\perp} \beta^{\xi}$.

Proof: Let $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{M}$ and $\xi \in \mathfrak{G}$. If $\alpha \hat{\perp} \beta$ then $\alpha | \beta$ so that $\alpha^{\xi} | \beta^{\xi}$. Thus, $\alpha^{\xi} \perp \beta^{\xi}$ or $\alpha^{\xi} \perp \beta^{\xi}$. If $\alpha^{\xi} \hat{\perp} \beta^{\xi}$ then $\alpha^{\xi}\beta^{\xi} = \gamma \in \mathcal{G}$. So we have $\alpha\beta = \gamma^{\xi^{-1}} \in \mathcal{G}$ by the invariance of $\mathcal{G}$, which contradicts $\alpha \hat{\perp} \beta$. Hence, $\alpha^{\xi} \hat{\perp} \beta^{\xi}$.

Conversely, suppose that $\alpha^{\xi} \hat{\perp} \beta^{\xi}$. because $\alpha^{\xi} \in \mathcal{G}$ and $\beta^{\xi} \in \mathcal{M}$ by the invariance of $\mathcal{G}$ and $\mathcal{M}$ then $\alpha^{\xi} = \gamma$, $\beta^{\xi} = \delta$ imply that $\gamma \hat{\perp} \delta$ and $\gamma^{\xi^{-1}} = \alpha \hat{\perp} \beta = \delta^{\xi^{-1}}$ by the paragraph above. ■

3.1.2.3. *For each* $\xi \in \mathfrak{G}$, $\mathcal{P}^{\xi} = \mathcal{P}$.

Proof: because $\mathcal{P} = \mathcal{G} \cup \mathcal{M}$, the result follows from the invariance of $\mathcal{G}$ and $\mathcal{M}$. ■

3.1.3. Properties of $\mathfrak{X}$.

3.1.3.1. *There exists a point; that is,* $\mathfrak{X} \neq \varnothing$.

Proof: Let $\alpha \in \mathcal{G}$. By Axiom 11, there exists $\gamma, \beta \in \mathcal{G}$ such that α, β, γ are distinct and mutually perpendicular. By Axiom 4, $\alpha\beta\gamma = \alpha\delta \notin \mathcal{G}$, where $\delta = \beta\gamma \in \mathcal{M}$ and $\alpha|\delta$ as $\alpha|\beta,\gamma$. Thus, $\alpha \,\hat{\perp}\, \delta$ and $P = \alpha\delta$ is a point. ■

3.1.3.2. *If* $O|\alpha,\beta,\gamma$; $\alpha,\beta,\gamma \in \mathcal{G}$; $\alpha \perp \beta \perp \gamma \perp \alpha$; *then* $O = \alpha\beta\gamma$.

Proof: By the proof of 3.1.3.1 above, $P = \alpha\beta\gamma$ is a point and $\delta = \beta\gamma \in \mathcal{M}$ because $\beta \perp \gamma$ with $\beta,\gamma \in \mathcal{G}$. So we have $\alpha \,\hat{\perp}\, \delta$, $O|\delta$ because $O|\beta,\gamma$. This yields $A, O|\alpha,\delta$ with $A = \alpha\delta$ so that $A = O$ by Axiom 3. ■

3.1.3.3. *If* $A \in \mathfrak{X}$ *and* $\delta \in \mathcal{P}$, *then* $A \neq \delta$; *that is, a point does not equal a plane.*

Proof: Let $A = \alpha\beta$ with $\alpha \in \mathcal{G}$, $\beta \in \mathcal{M}$, and $\alpha \,\hat{\perp}\, \beta$. Suppose that $A = \alpha\beta = \delta \in \mathcal{P}$. If $\delta \in \mathcal{M}$ then we have $\alpha = \beta\delta \in \mathcal{G}$ with $\beta|\delta$ and $\beta,\delta \in \mathcal{M}$, which contradicts Axiom 5. If $\delta \in \mathcal{G}$, then $A = \alpha\beta \in \mathcal{G}$ and this contradicts the definition of a point. ■

3.1.3.4. *The elements of* $\mathfrak{X}$ *are involutory.*

Proof: Let $A \in \mathfrak{X}$, so that we may write $A = \alpha\beta$ with $\alpha \,\hat{\perp}\, \beta$. In particular, $\alpha\beta = \beta\alpha$ and $AA = \alpha\beta\alpha\beta = \alpha\alpha\beta\beta = 1_{\mathfrak{G}}$. ■

3.1.4. General Consequences of the Axioms

3.1.4.1. *If* $P|\alpha,\beta$ *and* $\alpha \,\hat{\perp}\, \beta$ *then* $P = \alpha\beta$ *and if* $P = \alpha\beta$ *then* $P|\alpha,\beta$ *and* $\alpha \,\hat{\perp}\, \beta$.

Proof: If $P|\alpha,\beta$ and $\alpha \,\hat{\perp}\, \beta$ then $\alpha\beta = A$ for some $A \in \mathfrak{X}$ and we have $A,P|\alpha,\beta$ with $\alpha \,\hat{\perp}\, \beta$. Thus, by Axiom 3, $A = P$.

If $P = \alpha\beta$, then by 3.1.3.4 $\alpha|\beta$. because $P \notin \mathcal{P}$ by 3.1.3.3 then $\alpha \,\hat{\perp}\, \beta$. Also $P\alpha = \beta$ and $P\beta = \alpha$ imply that the products $P\alpha$ and $P\beta$ are involutory so that $P|\alpha,\beta$. ■

3.1.4.2. *If* $P|\alpha$ *then* $P\alpha \in \mathcal{P}$; *that is,* $P\alpha$ *is a plane* β *and* $P|\beta$.

Proof: By Axiom 1 there exists β such that $P|\beta$ and $\beta \hat{\perp} \alpha$. Thus, $P|\alpha,\beta$ with $\alpha \hat{\perp} \beta$ so by 3.1.4.1 above, $P = \alpha\beta$ and $\beta = P\alpha$. ■

3.1.4.3. *For each* $P \in \mathfrak{X}$ *and each* $\xi \in \mathfrak{G}$, $P^{\xi} \in \mathfrak{X}$.

Proof: By the definition of a point we may write $P = \alpha\beta$ with $\alpha \in \mathcal{M}$, $\beta \in \mathcal{G}$, and $\alpha \hat{\perp} \beta$. By 3.1.2.2, $\alpha^{\xi} \hat{\perp} \beta^{\xi}$ and $P^{\xi}|\alpha^{\xi},\beta^{\xi}$ so that $P^{\xi} = \alpha^{\xi}\beta^{\xi}$ is a point. ■

3.1.4.4. *If* $P|\alpha,\beta$ *and* $\gamma \hat{\perp} \alpha,\beta$ *then* $\alpha = \beta$. *(Given a point P and a plane* γ *there exists a unique plane* α *such that* $P|\alpha$ *and* $\alpha \hat{\perp} \gamma$.)

Proof: This follows immediately from Axiom 1. ■

3.1.4.5. *There do not exist three planes pairwise absolutely perpendicular.*

Proof: Suppose that α, β, and γ are pairwise absolutely perpendicular. Then for $P = \alpha\beta$ we have $P|\alpha,\beta$ with $\alpha,\beta \hat{\perp} \gamma$ so that $\alpha = \beta$, which contradicts $\alpha \hat{\perp} \beta$. ■

3.1.4.6. *If* $A,B,C|\alpha$ *then* $ABC = D|\alpha$.

Proof: By Axiom 8, ABC is a point D and $D\alpha = ABC\alpha = \alpha ABC = \alpha D$. ■

3.1.4.7. *If* $A,B,C|\alpha,\beta$ *then* $ABC|\alpha,\beta$.

Remark. From 3.1.4.6 and 3.1.4.7 above the product of three coplanar (and as we shall see, collinear) points yields a point which is coplanar (collinear) with the other three.

3.1.4.8. *If* $\alpha_1,\alpha_2,\alpha_3 \hat{\perp} \alpha$ *then* $\alpha_1\alpha_2\alpha_3 = \alpha_4 \in \mathcal{P}$ *and* $\alpha_4 \hat{\perp} \alpha$.

Proof: Let $A = \alpha\alpha_1$, $B = \alpha\alpha_2$, and $C = \alpha\alpha_3$. Then by Axiom 8, ABC is a point D and $D = \alpha\alpha_1\alpha\alpha_2\alpha\alpha_3 = \alpha\alpha_1\alpha_2\alpha_3$. So $D|\alpha$, by 3.1.4.6. By 3.1.4.2, $\alpha_1\alpha_2\alpha_3 = D\alpha = \alpha_4 \in \mathcal{P}$, $D = \alpha\alpha_4$, and $\alpha_1\alpha_2\alpha_3 = \alpha_4 \hat{\perp} \alpha$. ■

3.1.5. Perpendicular Plane Theorems

3.1.5.1. *If* $\alpha \perp \beta$; $\beta\gamma = A$ *and* $A|\alpha$ *then* $\alpha \perp \gamma$. *(A plane perpendicular to one of two absolutely perpendicular planes, and passing through their point of intersection, is perpendicular to both.)*

Proof: From our assumptions above it follows that $\alpha\gamma = \alpha\beta A = \beta A\alpha = \gamma\alpha$, so $\alpha \hat{\perp} \gamma$

or $\alpha \perp \gamma$ (note that $\alpha = \gamma$ implies that $\alpha \hat{\perp} \beta$ because $A = \beta\gamma$ and $A|\alpha$). Suppose that $\alpha \hat{\perp} \gamma$ so that $B = \alpha\gamma$. Then we have $A, B|\alpha,\gamma$ with $\alpha \hat{\perp} \gamma$, which implies that $A = B$ by Axiom 3. But if $A = B$ then $\beta\gamma = \alpha\gamma$ and $\beta = \alpha$, which contradicts $\beta \perp \alpha$. Thus, $\alpha\gamma$ is a plane and $\alpha \hat{\perp} \gamma$. ■

3.1.5.2. *Suppose that* $\alpha \perp \beta$; $A|\alpha,\beta,\gamma,\delta$; $\gamma\hat{\perp}\alpha$; *and* $\delta\hat{\perp}\beta$. *Then* $\delta \perp \gamma$. *(If two planes are perpendicular, their absolutely perpendicular planes at any point of their intersection are perpendicular.)*

Proof: By 3.1.4.1, $A = \gamma\alpha = \delta\beta$ and $\delta\gamma = \alpha\beta \in \mathcal{P}$ as $\alpha \perp \beta$. Thus, $\delta \perp \gamma$. ■

3.1.5.3. *If* $O = \alpha\alpha_1 = \gamma\gamma_1 = \varepsilon\varepsilon_1$ *with* $\alpha,\gamma,\varepsilon \in \mathcal{M}$ *and* $\alpha \perp \gamma \perp \varepsilon \perp \alpha$ *then* $\alpha\gamma = \varepsilon$.

Proof: Because $\alpha,\gamma,\varepsilon \in \mathcal{M}$, then $\alpha_1,\gamma_1,\varepsilon_1 \in \mathcal{G}$. Because $\alpha \perp \gamma \perp \varepsilon \perp \alpha$ and $\alpha\alpha_1 = \gamma\gamma_1 = \varepsilon\varepsilon_1$, then $\gamma\alpha = \gamma_1\alpha_1$; $\varepsilon\alpha = \varepsilon_1\alpha_1$; $\varepsilon\gamma = \varepsilon_1\gamma_1$ imply that $\gamma_1 \perp \alpha_1 \perp \varepsilon_1 \perp \gamma_1$. Because $O|\alpha_1,\gamma_1,\varepsilon_1$, then $O = \alpha_1\gamma_1\varepsilon_1$ by 3.1.3.2. Because points are involutions by 3.1.3.4, $1_{\mathfrak{G}} = OO = O\alpha_1\gamma_1\varepsilon_1 = O\alpha_1\gamma_1 OO\varepsilon_1 = \alpha\gamma\varepsilon$ and $\alpha\gamma = \varepsilon$. ■

3.1.6. Parallel Planes.

We say that two planes α and β are *parallel*, denoted by $\alpha \parallel \beta$, if $\alpha = \beta$, or there exists a γ such that $\alpha,\beta\hat{\perp}\gamma$.

3.1.6.1. *Parallelism is an equivalence relation on the set of planes* $\mathcal{P}$.

Proof: That the relation is reflexive and symmetric is clear. For transitivity suppose that $\alpha \parallel \beta$ and $\beta \parallel \gamma$ where α,β, and γ are distinct. Then there exists $\delta,\varepsilon \in \mathcal{P}$ such that $\alpha,\beta\hat{\perp}\delta$ and $\beta,\gamma\hat{\perp}\varepsilon$. Let $A = \alpha\delta$, $B = \beta\delta$, and $C = \beta\varepsilon$. Then by Axiom 8 we have $D = ABC = \alpha\delta\delta\beta\beta\varepsilon = \alpha\varepsilon$ and $\alpha\hat{\perp}\varepsilon$ so that $\alpha \parallel \gamma$. ■

3.1.6.2 *If* $\alpha,\beta\hat{\perp}\delta$ *and* $\beta\hat{\perp}\varepsilon$ *then* $\alpha\hat{\perp}\varepsilon$.

3.1.6.3. *If* $\alpha\hat{\perp}\gamma$, $\beta\hat{\perp}\delta$, *and* $\gamma \parallel \delta$ *then* $\alpha \parallel \beta$. *(Two planes absolutely perpendicular to two parallel planes are parallel.)*

Proof: If $\gamma = \delta$ then we have $\alpha,\beta\hat{\perp}\delta$ and the result follows. So assume that there is an ε such that $\gamma,\delta\hat{\perp}\varepsilon$. Then by 3.1.6.2 above, $\alpha\hat{\perp}\delta$, $\beta\hat{\perp}\gamma$, $\alpha,\beta\hat{\perp}\varepsilon$, and $\alpha \parallel \beta$. ■

3.1.6.4. *If* $\alpha \parallel \beta$, $\gamma \parallel \delta$, *and* $\beta \hat{\perp} \delta$ *then* $\alpha \hat{\perp} \gamma$. *(Two planes parallel to two absolutely perpendicular planes are absolutely perpendicular.)*

Proof: because $\alpha \parallel \beta$ then $\alpha, \beta \hat{\perp} \varepsilon$ for some ε and because $\beta \hat{\perp} \delta$ then $\alpha \hat{\perp} \delta$. Similarly, $\gamma, \delta \hat{\perp} \varepsilon'$ for some ε' and $\delta \hat{\perp} \beta$ imply that $\gamma \hat{\perp} \beta$. Hence, $\alpha, \beta \hat{\perp} \delta$ and $\beta \hat{\perp} \gamma$ yield $\alpha \hat{\perp} \gamma$. ■

3.1.6.5. *If* $\alpha \in \mathcal{M}(\mathcal{G})$ *and* $\alpha \parallel \beta$ *then* $\beta \in \mathcal{G}(\mathcal{M})$.

Proof: Suppose that $\alpha \in \mathcal{G}$ and $\alpha \parallel \beta$, so $\alpha, \beta \hat{\perp} \gamma$ for some γ. Because $\alpha \in \mathcal{G}$ then $\gamma \in \mathcal{M}$, which implies that $\beta \in \mathcal{G}$. ■

3.1.6.6. *If* $\alpha \parallel \beta$ *then* $\alpha^{\xi} \parallel \beta^{\xi}$ *for every* $\xi \in \mathfrak{G}$.

Proof: Let $\alpha, \beta \hat{\perp} \gamma$ for some $\gamma \in \mathcal{P}$. By 3.1.2.2, $\alpha^{\xi}, \beta^{\xi} \hat{\perp} \gamma^{\xi}$ so that $\alpha^{\xi} \parallel \beta^{\xi}$. From 3.1.6.5 and 3.1.6.6 above and the invariance of $\mathcal{G}$ and $\mathcal{M}$, we have that if $\alpha \parallel \beta$ and $\alpha \in \mathcal{G}(\mathcal{M})$ then $\beta \in \mathcal{G}(\mathcal{M})$ and $\alpha^{\xi} \parallel \beta^{\xi}$ with $\alpha^{\xi}, \beta^{\xi} \in \mathcal{G}(\mathcal{M})$. ■

3.1.6.7. *If* $\alpha \parallel \beta$ *then* $\mathfrak{X}_{\alpha} \cap \mathfrak{X}_{\beta} = \varnothing$ *or* $\alpha = \beta$.

Proof: Suppose that $\alpha, \beta \hat{\perp} \gamma$ and $P | \alpha, \beta$. Then $\alpha = \beta$ by 3.1.4.4. ■

3.1.6.8. *If* $A \neq B$ *then* $A \not\mid B$; *that is,* $AB \neq BA$.

Proof: By Axiom 6 there exists $\alpha \in \mathcal{P}$ such that $A, B | \alpha$. By 3.1.4.1 and 3.1.4.2 we may write $A = \alpha\alpha_1$ and $B = \alpha\alpha_2$. Suppose that $AB = BA$. Then we have $\alpha_1\alpha_2 = \alpha_2\alpha_1$ so that $\alpha_1 \hat{\perp} \alpha_2$ or $\alpha_1 \perp \alpha_2$. But $\alpha_1 \parallel \alpha_2$ because $\alpha_1, \alpha_2 \hat{\perp} \alpha$ so that either $\alpha_1 = \alpha_2$ or $\mathfrak{X}_{\alpha_1} \cap \mathfrak{X}_{\alpha_2} = \varnothing$. If $\alpha_1 \perp \alpha_2$ then by Axiom 7, $\mathfrak{X}_{\alpha} \cap \mathfrak{X}_{\beta} \neq \varnothing$ so $\alpha_1 = \alpha_2$ and $A = B$. If $\alpha_1 \hat{\perp} \alpha_2$ then $C = \alpha_1\alpha_2 \in \mathfrak{X}_{\alpha_1} \cap \mathfrak{X}_{\alpha_2}$ and again we have that $\alpha_1 = \alpha_2$. Hence, $A \neq B$. ■

3.1.6.9.a. *If* $A^B = A$, *then* $B = A$.

b. *If* $A \neq B$, *then* A, B, *and* A^B *are pairwise distinct.* ■

3.1.6.10. *If* $A | \alpha$; $B | \beta$; $C | \gamma$; *and* $\alpha \parallel \beta \parallel \gamma$ *then* $ABC | \alpha\beta\gamma$.

Proof: Let $A\alpha = \alpha'$, $B\beta = \beta'$, $C\gamma = \gamma'$. Then $\alpha', \beta', \gamma' \hat{\perp} \alpha, \beta, \gamma$ and $ABC = \alpha\alpha'\beta\beta'\gamma\gamma' = \alpha'\beta'\gamma' = \delta'\delta$ with $\alpha'\beta'\gamma' = \delta' \hat{\perp} \delta = \alpha\beta\gamma$. ■

Conclusion: $A | \alpha$; $B | \beta$; $\alpha \parallel \beta$, imply that $B' | \beta$.

3.1.6.11. *If* $AA' = BB'$; $A,A'|\alpha$; $B|\beta$; $\alpha \parallel \beta$ *then* $B'|\beta$.

Proof: Because $AA' = BB'$ then $B' = BAA'$; $\beta \parallel \alpha \parallel \alpha$. From 3.1.6.10 above, $B' = BAA'|\beta\alpha\alpha = \beta$. ■

3.1.6.12. *For each* P *and each* β *there is a unique* α *such that* $P|\alpha$ *and* $\alpha \parallel \beta$.

Proof: By Axiom 1 there exists a unique γ such that $\gamma\hat{\perp}\beta$. because $P|\gamma$ then $P\gamma = \alpha$, $\alpha\hat{\perp}\gamma$ and $\alpha \parallel \beta$. Now if $P|\delta$ and $\delta \parallel \beta$ then $P|\delta,\alpha$, so that $\alpha = \delta$ by 3.1.6.7. ■

3.1.6.12. *Let* $\alpha,\beta \in \mathcal{P}$ *and* $M \in \mathfrak{X}$. *Then* $\alpha\hat{\perp}\beta \leftrightarrow \alpha^M\hat{\perp}\beta$ *and thus,* $\alpha \parallel \alpha^M$.

Proof: Suppose that $\alpha\hat{\perp}\beta$. By Axiom 1 there exists unique $\gamma,\delta|M$ such that $\alpha\hat{\perp}\gamma$ and $\delta\hat{\perp}\beta$. By 3.1.6.2 we have $\beta,\gamma\hat{\perp}\alpha$ and $\beta\hat{\perp}\delta$ which implies that $\gamma\hat{\perp}\delta$. By 3.1.4.1, $M = \gamma\delta$. thus $\alpha^M = \alpha^{\gamma\delta} = \alpha^\delta\hat{\perp}\beta^\delta = \beta$. Therefore, $\alpha^M\hat{\perp}\beta$ and $\alpha^M \parallel \alpha$.

Conversely, suppose that $\alpha^M\hat{\perp}\beta$. As above, there exists unique $\gamma,\delta|M$ such that $\alpha^M\hat{\perp}\gamma$ and $\delta\hat{\perp}\beta$. It follows that $\beta\hat{\perp}\delta$, $M = \gamma\delta$, and $\alpha^\delta = \alpha^{\gamma\delta} = \alpha^M\hat{\perp}\beta$ and $\alpha = (\alpha^\delta)^\delta\hat{\perp}\beta^\delta = \beta$. ■

If $A^M = B$, then we say that M is the *midpoint* of A and B. Clearly, M is also the midpoint of B and A.

3.1.6.14. *If* $P^A = P^B$ *then* $A = B$. *(Uniqueness of midpoints.)*

Proof: From $P^A = P^B$ we have $PAB = ABP$ and also $PAB = PBA$ because ABP is a point and hence, an involution. Thus, $AB = BA$ which implies that $A = B$ by 3.1.6.8. ■

3.1.6.15. *If* $A,B|\alpha,\beta$ *and* $A^M = B$ *then* $M|\alpha,\beta$.

Proof: By Axiom 6 *there exists* γ,δ such that $A,M|\gamma,\delta$. because $B = A^M = MAM$ then by 3.1.4.7 we have $B|\gamma,\delta$. Thus, $A,B|\alpha,\gamma,\delta$; $M|\gamma,\delta$; and $A,B|\beta,\gamma,\delta$ so by Axiom2, $M|\beta,\alpha$. ■

3.1.6.16. .*If* $A,B,C|\delta$; $\alpha,\beta,\gamma\perp\delta$; $A|\alpha$; $B|\beta$; $C|\gamma$; $\alpha \parallel \beta \parallel \gamma$ *then* $ABC|\alpha\beta\gamma$ *and* $\alpha\beta\gamma \perp \delta$.

Proof: From 3.1.6.10 we know that $D = ABC|\alpha\beta\gamma = \varepsilon$. Because $\delta|\alpha,\beta,\gamma$ then $\varepsilon|\delta$ so that $\varepsilon \perp \delta$ or $\varepsilon\hat{\perp}\delta$. If $E = \delta\varepsilon$ then we have $D,E|\delta,\varepsilon$ with $\delta\hat{\perp}\varepsilon$ which implies that

$D = E$ by Axiom 3. Let $A = \alpha\alpha'$, $B = \beta\beta'$, and $C = \gamma\gamma'$. Because $\alpha \parallel \beta \parallel \gamma$ then $\alpha' \parallel \beta' \parallel \gamma'$. Thus, $\varepsilon\delta = D = ABC = \alpha\beta\gamma\alpha'\beta'\gamma' = \varepsilon\alpha'\beta'\gamma'$ which implies that $\delta = \alpha'\beta'\gamma'$. Because $A|\delta$ then we may write $A = \alpha\alpha' = \delta\delta'$ and by 3.1.5.1 we have $A|\alpha',\alpha,\delta$; $\alpha\hat{\perp}\alpha'$; and $\alpha \perp \delta$ which implies that $\alpha' \perp \delta$. Hence, $\alpha'\delta = \alpha'\alpha'\beta'\gamma' = \beta'\gamma'$ is a plane, so $\beta' \perp \gamma'$. By Axiom 7 and 3.1.6.7 we have $\beta' = \gamma'$ so that $\alpha' = \delta$. But this yields $\alpha\hat{\perp}\delta$ and $\alpha \perp \delta$, which is a contradiction. Thus, $\alpha\beta\gamma \perp \delta$. ■

3.1.6.17. *If* $A,B|\alpha$; $A|\alpha'$; $\alpha' \perp \alpha$; $B|\beta$; *and* $\beta \parallel \alpha'$ *then* $\beta \perp \alpha$.

Proof: By 3.1.6.16 above $B = AAB|\alpha'\alpha'\beta = \beta$ and $\beta \perp \alpha$. ■

3.1.6.18. *If* $E|\alpha,\varepsilon$; $\alpha \perp \beta$, *and* $\varepsilon\hat{\perp}\beta$ *then* $\varepsilon \perp \alpha$.

Proof: If $E|\beta$ then the result follows from 3.1.6.7. Suppose that $E \not| \beta$ and let $M = \varepsilon\beta$. Then, $E \neq E^{\beta}$ and $E^{\beta}|\alpha^{\beta} = \alpha$ so that $E, E^{\beta}|\alpha$. Now $E^{M} = E^{\varepsilon\beta} = E^{\beta}$, so that M is the midpoint of E and E^{β}. $E^{M} = E^{\beta\varepsilon}|\alpha^{\varepsilon}$ so that $E, E^{M}|\alpha,\alpha^{\varepsilon}$ and by 3.1.6.15, $M|\alpha,\alpha^{\varepsilon}$. In particular, $M|\alpha$ and we have $M|\alpha,\beta,\varepsilon$; $\alpha \perp \beta$; and $\beta\hat{\perp}\varepsilon$ so that $\alpha \perp \varepsilon$ by 3.1.6.7. ■

3.1.7. Consequences of Axiom 11 and 3.1.6.18.

3.1.7.1. *If* $A|\alpha$; $\alpha \perp \beta$; *and* $\alpha,\beta \in \mathcal{G}$ *then there exists a* γ *in* $\mathcal{G}$ *such that* $A|\gamma$ *and* $\gamma \perp \beta,\alpha$.

Proof: Let $A|\delta$ with $\delta\hat{\perp}\beta$. Then $\delta \in \mathcal{M}$ and $A|\alpha,\delta$ with $\alpha \perp \beta$, $\beta\hat{\perp}\delta$ so that $\alpha \perp \delta$ by 3.1.6.18. Because $\delta \in \mathcal{M}$, $\delta \perp \alpha$, and $\alpha \in \mathcal{G}$ then $\varepsilon = \delta\alpha \in \mathcal{G}$, $A|\varepsilon$, and $\varepsilon \perp \alpha$. We claim that $\varepsilon \perp \beta$. Indeed, $\beta\varepsilon = \beta\delta\alpha = \delta\beta\alpha = \delta\alpha\beta = \varepsilon\beta$ with $\varepsilon,\beta \in \mathcal{G}$ so that $\varepsilon \perp \beta$ or $\varepsilon = \beta$. But $\varepsilon \neq \beta$ because then $\delta\alpha = \beta$ and $\alpha = \delta\beta = P$, as $\delta\hat{\perp}\beta$. This contradicts 3.1.3.3. ■

3.1.7.2. *Every point may be written as a product of three mutually perpendicular planes from* $\mathcal{G}$.

Proof: Let A be any point. Then by definition $A = \alpha\beta$ for some $\alpha \in \mathcal{G}$ and some $\beta \in \mathcal{M}$ with $\alpha\hat{\perp}\beta$. By Axiom 11 there exists $\gamma \in \mathcal{G}$ such that $\gamma \perp \alpha$. By 3.1.7.1 there

exists a $\delta \in \mathcal{G}$ such that $A|\delta$ and $\delta \perp \alpha,\gamma$. Again by 3.1.7.1 there exists an $\varepsilon \in \mathcal{G}$ such that $A|\varepsilon$ and $\varepsilon \perp \delta,\alpha$. Hence, $A|\varepsilon,\alpha,\delta$; $\varepsilon,\delta,\alpha \in \mathcal{G}$, and $\varepsilon \perp \delta \perp \alpha \perp \varepsilon$ which yields by 3.1.3.2, $A = \delta\alpha\varepsilon$. ■

We note that in this case, $\alpha\beta = A = \alpha\delta\varepsilon$ implies that $\beta = \delta\varepsilon$; that is, if $A|\beta$ and $\beta \in \mathcal{M}$ then there exists $\beta_1,\beta_2 \in \mathcal{G}$ such that $A|\beta_1,\beta_2$ and $\beta = \beta_1\beta_2$.

3.1.7.3. *If $A|\alpha$ then there is a β in $\mathcal{G}$ such that $A|\beta$ and $\beta \perp \alpha$.*

Proof: This follows directly from the proof of 3.1.7.2, for if $\alpha \in \mathcal{G}$ then we can find $\delta,\varepsilon \in \mathcal{G}$ such that $A|\delta,\varepsilon$ and $\delta,\varepsilon \perp \alpha$. If $\alpha \in \mathcal{M}$ then we can find $\alpha_1,\alpha_2 \in \mathcal{G}$ such that $A|\alpha_1,\alpha_2$ and $\alpha_1,\alpha_2 \perp \alpha$. That is if $A|\alpha$ then there exists $\beta_1,\beta_2 \in \mathcal{G}$ such that $A|\beta_1,\beta_2$ and $\beta_1 \perp \beta_2$. Moreover, if $\alpha \in \mathcal{M}$ then we can find β_1,β_2 such that $\alpha = \beta_1\beta_2$ and if $\alpha \in \mathcal{G}$ then we can find $\beta_1,\beta_2 \in \mathcal{G}$ such that $\alpha\hat{\perp}\beta_1\beta_2$. ■

3.1.7.4. *If $A|\alpha \in \mathcal{M}$ and $\beta \perp \alpha$ then there is a γ such that $A|\gamma$ and $\gamma \perp \beta,\alpha$.*

Proof: Let $\delta|A$ with $\delta\hat{\perp}\beta$. Then $\delta \perp \alpha$ by 3.1.6.18 and $A|\varepsilon = \alpha\delta$. Because $\alpha \perp \beta$ and $\delta\hat{\perp}\beta$ then $\varepsilon|\beta$. If $\varepsilon\hat{\perp}\beta$ then we would have $A|\delta,\varepsilon$ with $\delta,\varepsilon\hat{\perp}\beta$ so $\delta = \varepsilon$ and $\alpha = 1_{\mathfrak{G}}$. Thus, $\varepsilon \perp \beta$. We note that if $\beta \in \mathcal{G}$ then $\delta,\varepsilon \in \mathcal{M}$ and if $\beta \in \mathcal{M}$ then $\varepsilon \in \mathcal{G}$. ■

3.2. Lines and Planes

3.2.1. General Theorems and Definitions.

3.2.1.1. *For any $A,B \in \mathfrak{X}$ and any $a,b \in \mathcal{G}$, if $A,B \in a,b$ then $A = B$ or $a = b$. Hence, by Axiom 6, for all $A,B \in \mathfrak{X}$, there exists a unique $g \in \mathcal{G}$, such that $A,B \in g$.*

Proof: Let $a = [\alpha,\beta]$, $b = [\gamma,\delta]$, $A,B|\alpha,\beta,\gamma,\delta$, and suppose that $A \neq B$. Let $C \in a$, so that $C|\alpha,\beta$. By Axiom 2 it follows that $C|\gamma,\delta$ so $C \in b$ and $a \subset b$. Similarly, $b \subset a$ and $a = b$. ■

3.2.1.2. *Every line contains at least three points.*

Proof: By definition, every line $g = g_{AB}$ contains at least two points A and B. Let $g_{AB} = [\alpha,\beta]$ with $A,B|\alpha,\beta$. Then by Axiom 8 and 3.1.4.7, $A^B|\alpha,\beta$ and $A^B \in \mathfrak{X}$. $A^B \neq A$ by 3.1.6.9. ■

3.2.1.3. *Suppose that* $\alpha\beta = \gamma$.

a. *If* $\alpha\alpha_1 = \beta\beta_1$ *then* $\alpha_1 \perp \beta_1$ *and* $\alpha_1\beta_1 = \gamma$, *so* $\gamma \perp \alpha_1, \beta_1$.

b. *If* $\alpha\alpha_1 = \beta\beta_1 = \gamma\gamma_1$ *then* $\alpha \perp \beta_1$, $\alpha_1 \perp \beta$, $\gamma_1 \perp \alpha_1, \beta_1$, *and* $\alpha\beta_1 = \alpha_1\beta = \gamma$.

c. *If* $A \mid \alpha, \beta$ *then* $A \mid \gamma$.

d. *The line* $g = [\alpha, \beta] = [\beta, \gamma] = [\alpha, \gamma] \equiv [\alpha, \beta, \gamma]$ *is a nonisotropic line. Thus,* $\alpha, \beta,$ *and* γ *are three mutually perpendicular planes which intersect in a line.*

Proof: a. From $\alpha\alpha_1 = \beta\beta_1$ we have $\gamma = \beta\alpha = \beta_1\alpha_1$.

b. If $\beta\beta_1 = \gamma\gamma_1$ and $\alpha\alpha_1 = \gamma\gamma_1$ then $\gamma_1 = \gamma\beta\beta_1 = \alpha\beta_1$ and $\gamma_1 = \gamma\alpha\alpha_1 = \beta\alpha_1$.

c. Because $A \mid \alpha, \beta$ and $\alpha\beta = \gamma$ then $A\gamma = A\alpha\beta = \alpha\beta A = \gamma A$ and $A \mid \gamma$.

d. By Axiom 7 there exist points A and B such that $A, B \mid \alpha, \beta$ and by 3.2.1.3.c. above, $A, B \mid \gamma$. Thus, $A, B \in [\alpha, \beta], [\alpha, \gamma], [\beta, \gamma]$ and $[\alpha, \beta] = [\alpha, \gamma] = [\beta, \gamma] \equiv [\alpha, \beta, \gamma]$ is a nonisotropic line. ∎

3.2.1.4. *If A and B are collinear then A^ξ and B^ξ are collinear for any* $\xi \in \mathfrak{G}$.

Proof: This follows from the fact that $A, B \mid \alpha, \beta \leftrightarrow A^\xi, B^\xi \mid \alpha^\xi, \beta^\xi$. ∎

For each $\xi \in \mathfrak{G}$, we define $a^\xi \equiv \{C^\xi : C \in a\}$. By 3.2.1.4 above, a^ξ is a line for every ξ in $\mathfrak{G}$ and if $a = [\alpha, \beta]$ then $a^\xi = [\alpha^\xi, \beta^\xi]$.

Definition of parallel lines and planes. We say the *line a is parallel to the plane* α, denoted by $a \parallel \alpha$, if there exists a β such that $a \subseteq \mathfrak{X}_\beta$ and $\beta \parallel \alpha$; that is, $a = [\beta, \gamma]$ and $\alpha, \beta \perp \delta$ for some δ.

We say that *two lines a and b are parallel*, $a \parallel b$, if there exists $\alpha, \beta, \gamma, \delta$ such that $a = [\alpha, \gamma]$, $b = [\beta, \delta]$, where $\alpha \parallel \beta$ and $\gamma \parallel \delta$.

3.2.1.5. *If* $AA' = BB' \neq 1_{\mathfrak{G}}$; $A, A' \mid \alpha, \alpha'$; $\alpha \neq \alpha'$; $B, B' \mid \beta, \beta'$; $\beta \neq \beta'$, *then* $g_{AA'} \parallel g_{BB'}$.

Proof: Let $B \mid \beta^*$ with $\beta^* \parallel \alpha$ (3.1.6.12). Then $B' \mid \beta^*$ by 3.1.6.11. Let $B \mid \beta_1^*$, with $\beta_1^* \parallel \alpha_1$. Then $B' \mid \beta_1^*$ by 3.1.6.11 and $B, B' \mid \beta, \beta_1, \beta^*, \beta_1^*$ which implies that $[\beta, \beta_1] = [\beta^*, \beta_1^*] = g_{BB'}$ by 3.2.1.1. Hence, we have $g_{AA'} = [\alpha, \alpha_1]$, $g_{BB'} = [\beta^*, \beta_1^*]$ with $\alpha \parallel \beta^*$ and $\alpha_1 \parallel \beta_1$; thus, $g_{AA'} \parallel g_{BB'}$. ∎

3.2.1.6. *Two lines,* $a = [\alpha, \alpha_1]$ *and* $b = [\beta, \beta_1]$ *are parallel precisely when there exist* $A, A' \in a$ *and* $B, B' \in b$ *such that* $AA' = BB' \neq 1_{\mathfrak{G}}$.

Proof: Suppose that $a \parallel b$. Then, $\alpha \parallel \beta$ and $\alpha_1 \parallel \beta_1$. Let $A, A' \in a$ and $B \in b$, so that $B' = AA'B | \alpha\alpha\beta$, and $\alpha_1\alpha_1\beta_1 = \beta, \beta_1$, by 3.1.6.10. Thus, $B, B' \in b$ with $AA' = BB'$. If $AA' = BB' = 1_{\mathfrak{G}}$ then $A = A'$. ■

3.2.1.7. *If* $a \parallel b$ *and* $b \parallel c$ *then* $a \parallel c$.

Proof: Let $a = [\alpha, \alpha']$. From the proof of 3.2.1.5 above we may write $b = [\beta, \beta']$ with $\alpha \parallel \beta$ and $\alpha' \parallel \beta'$ because $a \parallel b$, and $c = [\gamma, \gamma']$, where $\gamma \parallel \beta$ and $\gamma' \parallel \beta'$ because $b \parallel c$. By 3.1.6.1, $\alpha \parallel \gamma$ and $\alpha' \parallel \gamma'$ so that $a \parallel c$. ■

3.2.1.8. *For each line* a *and point* A *there is a unique line* b *such that* $A \in b$ *and* $b \parallel a$.

Proof: Let $B, C \in a$ be distinct and if $A \in a$ we can choose $B, C \neq A$ because every line contains at least three points by 3.2.1.2. Then $ABC = D$ by Axiom 10 and $BC = AD \neq 1_{\mathfrak{G}}$, so $g_{AD} \parallel a$ by 3.2.1.6. Now suppose that $A \in c$ and $c \parallel a$. By 3.2.1.7 above, $c \parallel g_{AD}$, so there exist $W, X \in c$ and $Y, Z \in g_{AD}$ such that $WX = YZ \neq 1_{\mathfrak{G}}$ by 3.2.1.6. By 3.1.4.7, $A' = AWX \in c$ and $A_1 = AYZ \in g_{AD}$. Thus, $1_{\mathfrak{G}} \neq AA' = WX = YZ = AA_1$ implies that $A' = A_1$. Hence, $A, A' \in g_{AD}, c$; $A \neq A'$, because $WX \neq 1_{\mathfrak{G}}$ and $g_{AD} = c$ by 3.2.1.1. ■

3.2.1.9. *If* $A, B \in g$, $A \neq B$, *and* $C \in h$ *then* $g \parallel h$ *if and only if* $ABC \in h$.

Proof: If $ABC = H \in h$, then $1_{\mathfrak{G}} \neq AB = HC$ and $g \parallel h$. Conversely, suppose that $g \parallel h$ and put $D = ABC$. Then because $A \neq B$,

$$1_{\mathfrak{G}} \neq AB = DC \text{ and } C \in g_{DC}, \text{ with } g_{DC} \parallel g.$$

Because $g \parallel h$ and $C \in h$, then by 3.2.1.8, $g_{DC} = h$ and $D \in h$. ■

3.2.1.10. *If* $a \parallel b$ *then* $a = b$ *or* $a \cap b = \varnothing$.

Proof: Suppose $a \parallel b$ and $a \cap b \neq \varnothing$. Let $A \in a, b$ and $C \in a$ with $C \neq A$, $B \in b$, with $A \neq B$. Then, $a = g_{AC}$, $b = g_{AB}$, and therefore, $D = ACB \in a, b$.

If $D = A$, then $ACB = A$, $C = B$, and $a = b$. If $D \neq A$, then it follows that $a = g_{AD} = g_{AC} = g_{BD} = g_{AB} = b$. ∎

Classification of nonisotropic lines. Following the terminology of physics we make the following definitions. Let a be a nonisotropic line. If there are elements $\alpha,\beta,\gamma \in \mathcal{M}$ such that $\alpha = \beta\gamma$ and $a = [\alpha,\beta,\gamma]$, then we say that a is *timelike*. If there are elements $\alpha,\beta \in \mathcal{G}$ such $\alpha \neq \beta$ and $a = [\alpha,\beta,\alpha b]$, then we say that a is *spacelike*.

Remark. Let $A,B|\alpha \in \mathcal{G}$ with $A \neq B$. Then by Axiom 14, there is a β such that $A,B|\beta$ and $\beta \perp \alpha$. Thus, $a = [\alpha,\beta]$ is a spacelike line by definition, $A,B \in a$, and every pair of points in a spacelike plane is joinable with a spacelike line.

3.2.2. Isotropic Lines.

3.2.2.1. *If $A \neq B$; $A,B|\alpha,\beta$ with $\beta \perp \alpha$, $\alpha \in \mathcal{M}$; and A and B are unjoinable with P; $P|\alpha,\gamma$; and $\gamma \perp \alpha,\beta$, then $A^\gamma = B$.*

Proof: First, $A^\gamma|\alpha^\gamma,\beta^\gamma = \alpha,\beta$. If $A^\gamma = A$ then $A|\gamma$ and A is unjoinable with P. Similarly, $B^\gamma|\alpha,\beta$ and $B^\gamma \neq B$. Suppose that A^γ and P are joinable. Then there are δ and ε such that $P,A^\gamma|\delta,\varepsilon$ and $\delta \perp \varepsilon$. But then $P,A = P^\gamma,(A^\gamma)^\gamma|\delta^\gamma,\varepsilon^\gamma$ and $\delta^\gamma \perp \varepsilon^\gamma$, which implies that P and A are joinable. Thus, A^γ and P are unjoinable, so by Axiom 13, A^γ is unjoinable with A or A^γ is unjoinable with B or $A^\gamma = A$ or $A^\gamma = B$. Hence, $A^\gamma = B$. ∎

3.2.2.2. *Suppose that $A \neq B$; $A,B|\alpha,\beta$ with $\alpha \perp \beta$, $\alpha \in \mathcal{M}$; A and B are unjoinable with $P|\alpha$; $P|\gamma$, $\gamma \hat{\perp} \beta$. then $A^\gamma = B$.*

Proof: First observe that because $P|\alpha,\gamma$; $\alpha \perp \beta$; and $\gamma\hat{\perp}\beta$, then $\alpha \perp \gamma$ by 3.1.6.18. Then $A^\gamma|\alpha^\gamma$, $\beta^\gamma = \alpha,\beta$ and A^γ is joinable with B. $A^\gamma \neq A$ because A and P are unjoinable as in 3.2.2.1 above. A^γ joinable with P implies that $A = (A^\gamma)^\gamma$ is joinable with $P^\gamma = P$. Hence, by Axiom 13, $A^\gamma = B$. ∎

3.2.2.3. *If A and B are unjoinable then $X \in g_{AB}$ precisely when $X = A$ or $X = B$ or X is unjoinable with both A and B.*

Proof: By Axiom 6, $g_{AB} = [\alpha, \beta]$ for some α and β in $\mathcal{P}$. Let $X \in g_{AB}$ and suppose that $X \neq A, B$. If X is joinable with A then there exists γ, δ such that $\gamma \perp \delta$ and $A, X | \gamma, \delta$. By Axiom 2, $B | \gamma, \delta$ and A and B are joinable. Hence, X is unjoinable with both A and B. ■

Remark. If X is unjoinable with A and B; $A, B | \alpha, \beta$; then by Axiom 10, $X | \alpha, \beta$ and $X \in g_{AB}$.

3.2.2.4. *If g_{AB} is isotropic; $C, D \in g_{AB}$; and $C \neq D$, then C and D are unjoinable.*

Proof: If $C, D | \alpha, \beta$ with $\alpha \perp \beta$ then $A, B | \alpha, \beta$ by Axiom 2; that is, $g_{AB} = g_{CD}$. ■

3.2.2.5. *If $A, B, C | \alpha$ are pairwise joinable and distinct, then each $P | \alpha$ is joinable with at least one of $A, B,$ and C.*

Proof: If $\alpha \in \mathcal{G}$ then the result follows from Axiom 12, so assume that $\alpha \in \mathcal{M}$. By Axiom 13 there exist $D, E | \alpha$ such that $A, B,$ and C are all unjoinable with P. Then by Axiom 13, at least two of $A, B,$ and C must lie on one of g_{PD} and g_{PE}. By 3.2.2.4 above, this implies that two of $A, B,$ and C are unjoinable, which contradicts our assumption. ■

3.2.2.6. *If g_{AB} is isotropic then g_{AB}^{ξ} is isotropic for all $\xi \in \mathfrak{G}$.*

Proof: If $A^{\xi}, B^{\xi} | \alpha, \beta$ with $\alpha \perp \beta$ then $A, B | \alpha^{\xi^{-1}}, \beta^{\xi^{-1}}$ and $\alpha^{\xi^{-1}} \perp \beta^{\xi^{-1}}$. ■

3.2.2.7. *If $P | \alpha \in \mathcal{M}$ then there are at most two isotropic lines in $\mathfrak{X}_{\alpha}$ through P.*

Proof: If $g_{AP}, g_{BP}, g_{CP} \subset \mathfrak{X}_{\alpha}$ are three distinct isotropic lines through P, then P is unjoinable with $A, B,$ and C. The points A, B, C must be pairwise distinct and are joinable by Axiom 13, in contradiction to 3.2.2.5. ■

3.2.2.8. *By Axiom 13 and 3.2.2.7, for each $P | \alpha$, $\alpha \in \mathcal{M}$, there are precisely two isotropic lines in $\mathfrak{X}_{\alpha}$ through P.*

3.2.2.9. *Let $g_{AB}, g_{AC} \subset \mathfrak{X}_{\alpha}$, $\alpha \in \mathcal{M}$, be two isotropic lines.*

a. *If $A | \beta$, with $\beta \hat{\perp} \alpha$, then $g_{AB}^{\beta} = g_{AC}$.*

b. *If $A | \gamma$, with $\gamma \hat{\perp} \alpha$, then $g_{AB}^{\gamma} = g_{AC}$.*

Proof: Because $g_{AB} \neq g_{AC}$ by assumption, then B is joinable with C by Axiom 13. The results follow from 3.2.2.1 and 3.2.2.2. ■

3.2.2.10. *If* $A,B,C|\alpha$, $\alpha \in \mathcal{M}$; $g_{AB} \neq g_{AC}$ *are isotropic then* g_{BC} *is nonisotropic and there is a* β *with* $\beta|A$ *such that* $\beta \perp \alpha$ *and* $C = B^{\beta}$.

Proof: Because $g_{AB} \neq g_{AC}$, then g_{BC} is nonisotropic. By Axiom 14 there exists a γ, $\gamma|B,C$, such that $\gamma \perp \alpha$. Let $A|\beta$, $\beta \not\perp \gamma$ (Axiom 1). Because $A|\alpha,\beta$; $\alpha \perp \gamma$; and $\beta \not\perp \gamma$, then by 3.1.6.18, $\alpha \perp \beta$. If B^{β}=B then $B|\beta$ and g_{AB} is nonisotropic. Similarly, $C^{\beta} \neq C$ and by 3.2.2.1, $B^{\beta} = C$. ■

3.2.2.11. *If* $A,B,C|\alpha$, $\alpha \in \mathcal{M}$, *are pairwise unjoinable then there exist* $\beta,\gamma \perp \alpha$ *with* $A|\beta,\gamma$ *such that* $C = B^{\beta\gamma}$.

Proof: First observe that by 3.2.2.3 and 3.2.2.4,

$$g_{AB} = g_{BC} = g_{AC} \subset \mathfrak{X}_{\alpha}$$

is isotropic. By 3.1.7.3 and 3.1.7.4, there exist $\beta|A$ such that $\beta \perp \alpha$. Because B and C are unjoinable with A, then $B,C \not| \beta$ and $B^{\beta} \neq B$. By Axiom 1 and 3.1.6.18, there is a $\delta|\beta$ such that $\delta \not\perp \beta$ and $\delta \perp \alpha$. Thus, $B,B^{\beta}|\delta,\delta^{\beta} = \delta,\alpha$ and B and B^{β} are joinable. Suppose that B^{β} and C are unjoinable. By 3.2.2.3, either $g_{B^{\beta}C} = g_{AC}$ or A is joinable with B^{β}. But if $A,B^{\beta}|\varepsilon$, $\varepsilon \perp \alpha$, then $A = A^{\beta},B|\varepsilon^{\beta} \perp \alpha^{\beta} = \alpha$ imply that A and B are joinable. If $g_{B^{\beta}C} = g_{AC} = g_{AB}$ then B^{β} is unjoinable with B. Hence, B^{β} and C are joinable and by 3.2.2.10, there is a γ with $\gamma|A$ such that $\gamma \perp \alpha$ and $C = B^{\beta\gamma}$. ■

3.2.2.12. *Let* $P|\alpha,\beta,\gamma$ *with* $\beta,\gamma \perp \alpha$, $\alpha \in \mathcal{M}$. *If* $g_{AB} \subset \mathfrak{X}_{\alpha}$ *is isotropic then* $g_{AB}^{\beta\gamma} \subset \mathfrak{X}_{\alpha}$ *is isotropic and* $g_{AB} \parallel g_{AB}^{\beta\gamma}$.

Proof: By 3.2.2.8, there are precisely two isotropic lines in $\mathfrak{X}_{\alpha}$ through P, say g_{PC} and g_{PQ}. By 3.2.2.9, $g_{PC}^{\beta} = g_{PQ}$ and $g_{PC}^{\beta\gamma} = g_{PC}$. Now $PAB = D$ is a point by Axiom 8, so $AB = PD$ and D is unjoinable with P as A is unjoinable with B. (Suppose that $P,D|\alpha'$ with $\alpha' \perp \alpha$. Let $B|\lambda$ with $\lambda \parallel \alpha'$. Then because $B|\alpha$, $\lambda \perp \alpha$ by 3.1.6.17. But then by 3.1.6.16, $A = PDB|\alpha'\alpha'\lambda = \lambda$ and A is joinable with B.) Because

$A,B,P|\alpha$ then $D = PAB|\alpha$. Hence, $D \in g_{PC}$ or $D \in g_{PQ}$, and $g_{AB} \parallel g_{PC}$ or $g_{AB} \parallel g_{PQ}$. thus $g_{AB}^{\beta\gamma} \parallel g_{PQ}^{\beta\gamma} = g_{PQ} \parallel g_{AB}$ or $g_{AB}^{\beta\gamma} \parallel g_{PC}^{\beta\gamma} = g_{PC} \parallel g_{AB}$. ■

3.3. A Reduction to Two Dimensions

In this section we restrict our attention to two dimensions. In this way we are able to use the work of Wolff [30] to construct our field.

Let ξ be any element in $\mathfrak{G}$; then the map $\sigma_\xi : \mathfrak{X} \mapsto \mathfrak{X}$ given by $\sigma_\xi(A) \equiv \xi A \xi^{-1}$ is bijective and maps lines onto lines and planes onto planes by 3.2.1.4,5,6 and 3.1.2.3. Hence, it is a collineation of $(\mathfrak{G}, \mathcal{G}, \mathfrak{X})$. In the following, for any element $\xi \in \mathfrak{G}$, the collineation induced by it is denoted by σ_ξ.

Let $\alpha \in \mathcal{P}$ be fixed throughout this section. We wish to define a set $\mathcal{C}_\alpha$ of maps on $\mathfrak{X}_\alpha$ which can be viewed as line reflections in a given plane $\mathfrak{X}_\alpha$. We then show that each element of $\mathcal{C}_\alpha$ is involutory and that $\mathcal{C}_\alpha$ forms an invariant system of generators within the group $\mathcal{G}_\alpha$ it generates. Finally we will show that $(\mathcal{C}_\alpha, \mathcal{G}_\alpha)$ satisfy Wolff's axioms [30] for a two dimensional Minkowski space; that is; the Lorentz plane, for $\alpha \in \mathcal{M}$.

Let $\mathcal{L}_\alpha = \{g \subset \mathfrak{X}_\alpha : g \text{ is nonisotropic}\}$. By Axiom 14, each $g \in \mathcal{L}_\alpha$ may be written uniquely in the form $g = [\alpha, \beta, \gamma]$ where $\alpha = \beta\gamma$. Let $A \in \mathfrak{X}_\alpha$ and $g = [\alpha, \beta, \gamma] \in \mathcal{L}_\alpha$. because $A|\alpha$ and $\alpha = \beta\gamma$, then $A = A^\alpha = A^{\beta\gamma}$ and $A^\beta = A^\gamma$. Hence, we define the map $\sigma_g : \mathfrak{X}_\alpha \mapsto \mathfrak{X}_\alpha$ by $\sigma_g(A) \equiv A^g \equiv A^\beta = A^\gamma$, for $A \in \mathfrak{X}_\alpha$, and $\sigma_g : \mathcal{L}_\alpha \mapsto \mathcal{L}_\alpha$ by $\sigma_g(h) \equiv h^g \equiv \{\sigma_g(A) : A \in h\}$. Note that by 3.2.1.4, if $h = [\alpha, \delta, \varepsilon]$, then $\sigma_g(h) = [\alpha, \delta^\beta, \varepsilon^\beta] = [\alpha, \delta^\gamma, \varepsilon^\gamma]$.

3.3.1. *For each* $g = [\alpha, \beta, \gamma] \in \mathcal{L}_\alpha$, $\sigma_g : \mathfrak{X}_\alpha \mapsto \mathfrak{X}_\alpha$ *is bijective.*

Proof: If $A,B|\alpha$ and $\sigma_g(A) = \sigma_g(B)$ then $A^\gamma = B^\gamma$ and $A = B$, so σ_g is injective. If $A|\alpha$ then $A^\gamma|\alpha^\gamma = \alpha$ because $\alpha \perp \gamma$ and $\sigma_g(A^\gamma) = (A^\gamma)^\gamma = A$, so σ_g is onto. ■

3.3.2. *For* $g = [\alpha, \beta, \gamma] \in \mathcal{L}_\alpha$, $\sigma_g : \mathcal{L}_\alpha \mapsto \mathcal{L}_\alpha$ *is bijective.*

Proof: Let $h = [\alpha, \varepsilon, \eta] \in \mathcal{L}_\alpha$. Because $\alpha \perp \gamma$ and $\varepsilon \perp \eta \leftrightarrow \varepsilon^\gamma \perp \eta^\gamma$ by 3.1.2.1 then

$\alpha = \alpha^\gamma = \varepsilon^\gamma\eta^\gamma$. It follows that $\sigma_g(h) = [\alpha, \varepsilon^\gamma, \eta^\gamma] \in \mathcal{L}_\alpha$. If $A|\varepsilon,\eta$ then $A^\gamma|\varepsilon^\gamma,\eta^\gamma$ so $\sigma_g(A) \in \sigma_g(h)$ for all $A \in h$ and σ_g is a well-defined collineation.

If $l = [\alpha, \lambda, \mu] \in \mathcal{L}_\alpha$ and $\sigma_g(h) = \sigma_g(l)$ then for every A in h we have

$$\sigma_g(A) = A^\gamma \in \sigma_g(l) = [\alpha, \lambda^\gamma, \mu^\gamma].$$

This implies that $A = (A^\gamma)^\gamma \in [\alpha, \lambda, \mu] = l$; that is, $h = l$. Hence, $\sigma_g : \mathcal{L}_\alpha \mapsto \mathcal{L}_\alpha$ is injective. Because $\varepsilon \perp \eta \leftrightarrow \varepsilon^\gamma \perp \eta^\gamma$ then it follows that

$$h = [\alpha, \varepsilon, \eta] \in \mathcal{L}_\alpha \leftrightarrow \sigma_g(h) = [\alpha, \varepsilon^\gamma, \eta^\gamma] \in \mathcal{L}_\alpha.$$

Hence, $\sigma_g : \mathcal{L}_\alpha \mapsto \mathcal{L}_\alpha$ is surjective. ■

3.3.3. *Each* σ_g, *for* $g \in \mathcal{L}_\alpha$, *is involutory.*

Proof: Let $g = [\alpha, \beta, \gamma]$. Then for $A|\alpha$ we have $\sigma_g\sigma_g(A) = \sigma_g(A^\gamma) = (A^\gamma)^\gamma = A$. ■

Let $g = [\alpha, \delta, \gamma]$, $h = [\alpha, \varepsilon, \eta] \in \mathcal{L}_\alpha$. We say that g is *perpendicular to or orthogonal to h*, denoted $g \perp h$, if one of γ and δ is absolutely perpendicular to one of ε and η and $g \cap h \neq \varnothing$.

3.3.4. *For g and h as above, if* $\gamma \hat{\perp} \varepsilon$ *then* $\delta \hat{\perp} \eta$, $\delta \perp \varepsilon$, *and* $\gamma \perp \varepsilon$. *Moreover,* $M = \gamma\varepsilon = \delta\eta \in g, h$.

Proof: Let $M = \gamma\varepsilon$. Because $\gamma, \varepsilon \perp \alpha$ then $M^\alpha = (\gamma\varepsilon)^\alpha = \gamma^\alpha\varepsilon^\alpha = \gamma\varepsilon = M$ and $M|\alpha$. So we have $M|\alpha,\varepsilon$ which implies that $M|\alpha\varepsilon = \eta$ and $M \in h$. Also, $M|\alpha,\gamma$ so $M|\alpha\gamma = \delta$ and $M \in g$. Also from $M = \gamma\varepsilon$ it follows that $M\varepsilon = \gamma = \delta\alpha = \delta\eta\varepsilon$ so that $M = \delta\eta$ and $\delta\hat{\perp}\eta$. Applying 3.1.6.18 to $M|\gamma,\eta$ with $\gamma \perp \delta$, $\eta\hat{\perp}\delta$, and $M|\delta,\varepsilon$ with $\delta \perp \gamma$, $\varepsilon\hat{\perp}\gamma$, we obtain $\gamma \perp \eta$ and $\delta \perp \varepsilon$. ■

3.3.5. *If* $a, b, g \in \mathcal{L}_\alpha$ *then* $a \perp b \leftrightarrow a^g \perp b^g$.

Proof: By 3.1.2.1 and 3.1.2.2 we have

$$a \subset \mathfrak{X}_\gamma;\ \ \gamma\hat{\perp}\delta;\ \ b \subset \mathfrak{X}_\delta \leftrightarrow a^\xi \subset \mathfrak{X}_\gamma^\xi;\ \ \gamma^\xi\hat{\perp}\delta^\xi;\ \ b^\xi \subset \mathfrak{X}_\delta^\xi, \text{ for all } \xi \in \mathfrak{G}.$$

The result follows. ■

Remark. Because $A|\varepsilon \leftrightarrow A^{\xi}|\varepsilon^{\xi}$ for all $\xi \in \mathfrak{G}$, $A \in \mathfrak{X}$, and $\varepsilon \in \mathcal{P}$, it follows that for each $g \in \mathcal{L}_\alpha$, σ_g maps:

(i) The set $\mathfrak{X}_\alpha$ onto itself.

(ii) The set $\mathcal{L}_\alpha$ onto itself.

(iii) Collinear points in $\mathfrak{X}_\alpha$ onto collinear points in $\mathfrak{X}_\alpha$.

(iv) orthogonal lines in $\mathfrak{X}_\alpha$ onto orthogonal lines in $\mathfrak{X}_\alpha$.

That is, σ_g is an orthogonal collineation of $\mathfrak{X}_\alpha$.

Let $\mathcal{C}_\alpha = \{\sigma_g : g \in \mathcal{L}_\alpha\}$ and $\mathfrak{P}_\alpha = \{\sigma_P : P \in \mathfrak{X}_\alpha\}$.

3.3.6. *The sets $\mathcal{L}_\alpha$ and $\mathfrak{X}_\alpha$ are nonempty. Hence, $\mathcal{C}_\alpha \neq \varnothing$ and $\mathfrak{P}_\alpha \neq \varnothing$.*

Proof: If $\alpha \in \mathcal{M}$ then we may write $\alpha = \alpha'\delta$ where $\alpha',\delta \in \mathcal{G}$ and $\alpha' \perp \delta$ by definition of $\mathcal{M}$. By Axiom 7, there exist A,B such that $A \neq B$ and $A,B|\alpha',\delta$. Thus, $A,B|\alpha'\delta = \alpha$ and $\mathfrak{X}_\alpha \neq \varnothing$. Moreover, $g_{AB} = [\alpha,\alpha',\delta] \in \mathcal{L}_\alpha$ and $\mathcal{L}_\alpha \neq \varnothing$. If $\alpha \in \mathcal{G}$ then by 3.1.3.1, there exist $A|\alpha$ and by 3.1.7.3, there is a β in $\mathcal{G}$ such that $A|\beta$ and $\beta \perp \alpha$. Thus, $A \in \mathfrak{X}_\alpha$ and $g = [\alpha,\beta,\alpha\beta] \in \mathcal{L}_\alpha$. ■

Note that from 3.3.3, $\mathcal{C}_\alpha$ consists of involutory elements and by 3.1.3.4, $\mathfrak{P}_\alpha$ consists of involutory elements.

3.3.7. *If $P \in g,h \in \mathcal{L}_\alpha$ and $g \perp h$ then $\sigma_p = \sigma_g\sigma_h = \sigma_h\sigma_g$.*

Proof: Let $g = [\alpha,\gamma,\delta]$ and $h = [\alpha,\varepsilon,\eta]$, and without loss of generality, assume that $\gamma\hat{\perp}\varepsilon$, so that $\delta\hat{\perp}\eta$. By 3.3.4, $P = \gamma\varepsilon = \delta\eta \in g,h$ and by 3.2.1.1, $\{P\} = g \cap h$. Then for $A \in \mathfrak{X}_\alpha$,

$$\sigma_g\sigma_h(A) = \sigma_g(A^{\varepsilon}) = A^{\varepsilon\gamma} = \sigma_P(A) = A^{\gamma\varepsilon} = \sigma_h(A^{\gamma}) = \sigma_h\sigma_g(A). \ ■$$

3.3.8. *For every A in $\mathfrak{X}_\alpha$ and for every $h \in \mathcal{L}_\alpha$, there is a unique $g \in \mathcal{L}_\alpha$ such that $A \in g$ and $g \perp h$.*

Proof: Let $h = [\alpha,\varepsilon,\eta] \in \mathcal{L}_\alpha$ and $A \in \mathfrak{X}_\alpha$. By Axiom 1, there exist a $\gamma|A$ such that $\gamma\hat{\perp}\varepsilon$. Thus, $A|\alpha,\gamma$ with $\alpha \perp \varepsilon$ and $\gamma\hat{\perp}\varepsilon$ so $\alpha \perp \gamma$ by 3.1.6.18. because $\alpha \perp \gamma$ then $\alpha\gamma = \delta$ and $A|\alpha,\gamma$ implies $A|\delta$ so that $A \in g = [\alpha,\gamma,\delta] \in \mathcal{L}_\alpha$. because $g \subset \mathfrak{X}_\gamma$, $\gamma\hat{\perp}\varepsilon$,

$A \in g = [\alpha,\gamma,\delta] \in \mathcal{L}_\alpha$. Because $g \subset \mathfrak{X}_\gamma$, $\gamma\hat{\perp}\varepsilon$, and $h \subset \mathfrak{X}_\varepsilon$, then $g \perp h$. By 3.3.4 above, $\eta\hat{\perp}\delta$ and $\{B\} = \{\gamma\varepsilon\} = \{\eta\delta\} = g \cap h$.

Now suppose that $l \subset \mathcal{L}_\alpha$ such that $A \in l$ and $l \perp h$. Now we may uniquely write $l = [\alpha,\alpha',\beta]$ in $\mathfrak{X}_\alpha$ where $\alpha = \alpha'\beta$. because $l \perp h$ then without loss of generality, we may assume that $\alpha'\hat{\perp}\varepsilon$ and $\beta\hat{\perp}\eta$. because $\gamma\hat{\perp}\varepsilon$ and $\delta\hat{\perp}\eta$ it follows that $\alpha' \parallel \gamma$ and $\beta \parallel \delta$. By the definition of parallel lines in Section 3.2 we have $l \parallel g$. because $A \in l \cap g$ then by 3.2.1.10, $l = g$. ■

3.3.9. (i) *The fixed points of* $\sigma_g \in \mathcal{C}_\alpha$ *are the points in* $\mathfrak{X}_\alpha$ *incident with* g.

(ii) *The fixed lines of* $\sigma_g \in \mathcal{C}_\alpha$ *consist of* g *and all lines in* $\mathcal{L}_\alpha$ *which are orthogonal to* g.

Proof: Let $g = [\alpha,\gamma,\delta]$. If $A = \sigma_g(A) = A^\gamma$ for $A|\alpha$, then $A|\gamma$. because $A^\gamma = A^\delta$, $\forall A|\alpha$ then $A^\delta = A$ and $A|\delta$. Thus, $A \in g$ and 3.3.9.(i) follows.

Let $h = [\alpha,\varepsilon,\eta] \in \mathcal{L}_\alpha$ and suppose that $\sigma_g(h) = h$; $g \neq h$. Then there is an $A \in h$ such that $A \notin g$. By 3.3.8, there is a $k = [\alpha,\omega,\theta] \in \mathcal{L}_\alpha$ such that $A \in k$ and $k \perp g$. Because $A \in h$, $\sigma_g(A) = A^\gamma \in h$ and because $A \in k$, $k \perp g$ then $A^\gamma|\omega^\gamma,\theta^\gamma = \omega,\theta$ and $A^\gamma \in k$. Because $A \notin g$ then $A \not| \gamma$ (for if $A|\gamma$ then $A|\alpha$ implies that $A|\alpha\gamma = \delta$ which implies that $A \in g$) and $A \neq A^\gamma$. Thus we have $A,A^\gamma \in h,k$, so that $h = k$ by 3.2.1.1. Hence, $h \perp g$. ■

3.3.10. (i) *The only fixed point of a point reflection,* σ_P, *is the point* P.

(ii) *The fixed lines of* σ_P *are the lines incident with* P.

Proof: If $P^A = P$, then this implies that $APA = A$, or $AP = 1_{\mathfrak{G}}$, or $A = P$. Thus, 3.3.10.(i) holds. Let $x = [\beta,\upsilon]$ be any line (not necessarily in $\mathfrak{X}_\alpha$ or nonisotropic) and $A \in x$. Then $P^A \in x$ implies that $P^A|\beta,\upsilon$, so that $P|\beta^A,\upsilon^A = \beta,\upsilon$, and $P \in x$. ■

Let $\mathcal{G}_\alpha$ be the group acting on $\mathfrak{X}_\alpha$ generated by $\mathcal{C}_\alpha$. By 3.3.5, every element of $\mathcal{G}_\alpha$ is an orthogonal collineation of $\mathfrak{X}_\alpha$. Let $\sigma \in \mathcal{G}_\alpha$. By a *transformation with* σ, we mean the adjoint action of σ on $\mathcal{G}_\alpha : \sigma_1 \in \mathcal{G}_\alpha \mapsto \sigma_1^\sigma \equiv \sigma\sigma_1\sigma^{-1} \in \mathcal{G}_\alpha$. Note that every such transformation is an inner automorphism of the group.

3.3.11. *Let* $h = [\alpha,\varepsilon,\eta]$, $g = [\alpha,\gamma,\delta] \in \mathcal{L}_\alpha$, *then* $\sigma_g^{\sigma_h} = \sigma_{\sigma_h(g)}$.

Proof: Let $A|\alpha$ and put $B = \sigma_g(A) = A^\gamma$. Then

$$\sigma_g^{\sigma_h}(\sigma_h(A)) \ = \sigma_h\sigma_g\sigma_h^{-1}\sigma_h(A) = \sigma_h\sigma_g(A) = \sigma_h(B).$$

Thus, $\sigma_g^{\sigma_h}(\sigma_h(A)) \ = \sigma_{\sigma_h(g)}(\sigma_h(A))$ for all $A \in \mathfrak{X}_\alpha$.Because σ_h is injective, then $\sigma_g^{\sigma_h} = \sigma_{\sigma_h(g)}$ for every A in $\mathfrak{X}_\alpha$. Similarly, $\sigma_g^{\sigma_h}(a) = \sigma_{\sigma_h(g)}(a)$, for every a in $\mathcal{L}_\alpha$. Hence, $\sigma_g^{\sigma_h} = \sigma_{\sigma_h(g)}$. Analogously, we can obtain $\sigma_P^{\sigma_h} = \sigma_{\sigma_h(P)}$, for every P in $\mathfrak{X}_\alpha$. ■

Because $\mathcal{C}_\alpha$ generates $\mathcal{G}_\alpha$ and every element of $\mathcal{G}_\alpha$ is an orthogonal collineation of $(\mathfrak{X}_\alpha,\mathcal{L}_\alpha)$ then we obtain the following.

3.3.12. *For every* $\sigma \in \mathcal{G}_\alpha$ *and for every* $\sigma_g \in \mathcal{C}_\alpha$, we have $\sigma_g^\sigma \in \mathcal{C}_\alpha$; that is, $\mathcal{C}_\alpha$ *is an invariant system of* $\mathcal{G}_\alpha$. ■

Consider the two mappings: $g \in \mathcal{L}_\alpha \mapsto \sigma_g \in \mathcal{C}_\alpha$ and $P \in \mathfrak{X}_\alpha \mapsto \sigma_P \in \mathfrak{P}_\alpha$. The first one is from the set of nonisotropic lines in $\mathcal{L}_\alpha$ onto the set of line reflections, and the second is from the set of points in $\mathfrak{X}_\alpha$ onto the set of point reflections in $\mathfrak{X}_\alpha$. These mappings are injective, because reflections in two distinct points have distinct sets of fixed points by 3.3.10, and similarly for lines and line reflections. Thus, it follows from 3.3.11, 3.3.12, and the preceding remark that the next result obtains.

3.3.13. *If* $\sigma \in \mathcal{G}_\alpha$ *and* $P,Q \in \mathfrak{X}_\alpha$ *then* $\sigma(P) = Q \leftrightarrow \sigma_P^\sigma = \sigma_Q$.

Proof: Because

$$\sigma(P) = Q \leftrightarrow \sigma_{\sigma(P)} = \sigma_Q \leftrightarrow \sigma_{\sigma(P)} = \sigma_P^\sigma,$$

the result follows. ■

3.3.14. *If* $\sigma \in \mathcal{G}_\alpha$ *and* $g,h \in \mathcal{L}_\alpha$ *then* $\sigma(g) = h \leftrightarrow \sigma_g^\sigma = \sigma_h$. ■

3.3.15. *If* $P \in \mathfrak{X}_\alpha$ *and* $g \in \mathcal{L}_\alpha$ *then* $P \in g \leftrightarrow \sigma_P\sigma_g$ *is involutory.*

Proof: Suppose that $P \in g = [\alpha,\gamma,\delta] \in \mathcal{L}_\alpha$. Then for $A|\alpha$,

$\sigma_P\sigma_g(A) = \sigma_P(A^\gamma) = A^{\gamma P} = A^{P\gamma} = \sigma_g\sigma_P(A)$. Thus, $(\sigma_P\sigma_g)^2 = 1_{\mathfrak{X}_\alpha}$. Now assume that

$\sigma_P\sigma_g$ is involutory. Then $P = \sigma_g\sigma_P\sigma_g\sigma_P(P) = P^{P\gamma P\gamma} = P^{\gamma P\gamma}$. Because $\gamma P\gamma = P^{\gamma} \in \mathfrak{X}_\alpha$ by 3.1.4.3, then by 3.1.6.9, $P = P^{\gamma}$. Because $P^{\gamma} = P^{\delta}$ then $P|\gamma,\delta$, and hence, $P \in g$. ∎

3.3.16. *If $g,h \in \mathcal{L}_\alpha$, then $g \perp h \leftrightarrow \sigma_g\sigma_h$ is involutory.*

Proof: Now $g \perp h$ if and only if $\sigma_g(h) = h$. For $g \neq h$, by 3.3.9, $\sigma_g(h) = h$ if and only if $\sigma_h^{\sigma_g} = \sigma_h$. For $\sigma_g \neq \sigma_h$ by 3.3.14, $\sigma_h^{\sigma_g} = \sigma_h$ if and only if $(\sigma_g\sigma_h)^2 = 1_{\mathfrak{X}_\alpha}$ and $\sigma_g\sigma_h \neq 1_{\mathfrak{X}_\alpha}$. ∎

3.3.17. *The point reflections $\mathfrak{P}_\alpha$ are the involutory products of two line reflections from $\mathcal{C}_\alpha$.*

Proof: Let $\sigma_P \in \mathfrak{P}_\alpha$ where $P \in \mathfrak{X}_\alpha$. Then we may write $P = \gamma\delta\varepsilon$ where $\alpha = \gamma\delta$ and $\gamma,\delta,\varepsilon \in \mathcal{G}$ by 3.1.7.3. Let $g = [\alpha,\gamma,\delta]$. Because $\alpha = \gamma\delta$, then $g \in \mathcal{L}_\alpha$. By 3.3.8, there is an l in $\mathcal{L}_\alpha$ such that $P \in l$ and $l \perp g$. By 3.3.7, $\sigma_P = \sigma_l\sigma_g = \sigma_g\sigma_l$. ∎

We now show that if $\alpha \in \mathcal{M}$, then the pair $(\mathcal{C}_\alpha,\mathcal{G}_\alpha)$, acting as maps on $(\mathfrak{X}_\alpha,\mathcal{L}_\alpha,\alpha)$, satisfies Wolff's axiom system [30] for his construction of two-dimensional Minkowski space. We give Wolff's axiom system below.

Basic assumption: A given group $\mathfrak{G}$, and its generating set $\mathcal{G}$ of involution elements, form invariant system $(\mathcal{G},\mathfrak{G})$

The elements of $\mathcal{G}$ will be denoted by lowercase Latin letters. Those involutory elements of $\mathfrak{G}$ that can be represented as the product of two elements in $\mathcal{G}$, ab with $a|b$, will be denoted by uppercase Latin letters.

Axiom 1. *For each P and for each g, there is an l with $P,g|l$.*

Axiom 2. *If $P,Q|g,l$ then $P = Q$ or $g = l$.*

Axiom 3. *If $P|a,b,c$ then $abc \in G$.*

Axiom 4. *If $g|a,b,c$ then $abc \in \mathcal{G}$.*

Axiom 5. *There exist Q,g,h such that $g|h$ but $Q \not| g,h,gh$.*

Axiom 6. *There exist A,B; $A \neq B$, such that $A,B \not| g$ for any $g \in \mathcal{G}$. (There exist unjoinable points.)*

Axiom 7. *For each P and A,B,C such that $A,B,C|g$, there is a $v \in \mathcal{G}$ such that $P,A|v$ or $P,B|v$ or $P,C|v$.*

Geometric meaning of the axioms. The elements denoted by small Latin letters (elements of $\mathcal{G}$) are called lines and those elements denoted by large Latin letters (thus the element ab with $a|b$), points. We say the point A and the line b are *incident* if $A|b$; the lines a and b are *perpendicular* (*orthogonal*), if $a|b$. Further we say two points A and B are *joinable* when there is a line g such that $A,B|g$.

Replacing $\mathcal{L}_\alpha$ with $\mathcal{C}_\alpha$ and $\mathfrak{X}_\alpha$ with $\mathfrak{P}_\alpha$, the pair $(\mathcal{C}_\alpha, \mathcal{G}_\alpha)$ satisfies the basic assumption by 3.3.3, 3.3.6, and 3.3.12. It follows from 3.3.15, 3.3.16, and 3.3.17, that our definition of points, our incidence relation, and our definition of orthogonality agree with those of Wolff. Hence, we may identify $\mathcal{L}_\alpha$ with $\mathcal{C}_\alpha$, $\mathcal{C}_\alpha$ with $\mathcal{G}$, and $\mathcal{G}_\alpha$ with $\mathfrak{G}$.

Verification of Wolff's axioms. Axiom 1 follows from 3.3.8 and Axiom 2 from 3.2.1.1. For Axiom 3, let $a = [\alpha,\alpha',\varepsilon]$, $b = [\alpha,\beta,\beta']$, $c = [\alpha,\gamma,\gamma'] \in \mathcal{G}$. Then $\alpha',\beta,\gamma \perp \alpha$, and by our Axiom 9, $\alpha'\beta\gamma = \delta \perp \alpha$ and $d = [\alpha,\delta,\alpha\delta] \in \mathcal{L}_\alpha$. For $A|\alpha$,

$$\sigma_a\sigma_b\sigma_c(A) = A^{\gamma\beta\alpha'} = A^{\alpha'\beta\gamma} = A^{\delta} = \sigma_d(A).$$

Hence, $\sigma_a\sigma_b\sigma_c = \sigma_d \in \mathcal{C}_\alpha$. If we then identify "|" with "$\in$", we get Wolff's Axiom 3. ■

Axiom 4. *If $a,b,c|g$, then $abc \in \mathcal{G}$.*

Proof: Let $g = [\alpha,\lambda,\lambda']$, $a = [\alpha,\alpha',\delta]$, $b = [\alpha,\beta,\beta']$, and $c = [\alpha,\gamma,\gamma']$, with $\alpha',\beta,\gamma \perp \lambda$. Then by 3.1.4.8, $\alpha'\beta\gamma = \varepsilon \perp \lambda$. Let $A \in a$, $B \in b$, and $C \in c$, then $A|\alpha',\alpha$; $B|\alpha,\beta$; and $C|\alpha,\gamma$. By 3.1.6.10 $ABC = D|\alpha'\beta\gamma = \varepsilon$ and $ABC = D|\alpha$ by 3.1.4.6. Thus, $D|\varepsilon,\alpha$; $\varepsilon \perp \lambda$; $\alpha \perp \lambda$, and $\varepsilon \perp \alpha$. So, $d = [\alpha,\varepsilon,\alpha\varepsilon] \in \mathcal{L}_\alpha$. And if $X|\alpha$, then $\sigma_a\sigma_b\sigma_c(X) = X^{\gamma\beta\alpha'} = X^{\varepsilon} = \sigma_d(X)$ and $\sigma_a\sigma_b\sigma_c = \sigma_d \in \mathcal{C}_\alpha$ ■

Axiom 5: *There exist Q,g,h such that $g|h$ but $Q \not| g,h,gh$.*

Proof: Because $\alpha \in \mathcal{M}$, we may write $\alpha = \beta\gamma$, where $\beta, \gamma \in \mathcal{G}$ and $\beta \perp \gamma$. By 3.2.1.2 every line contains at least three distinct points. Let $g = [\alpha, \beta, \gamma] \in \mathcal{L}_\alpha$ and let $B \in g$. By 3.3.8, there is an h in $\mathcal{L}_\alpha$ such that B in h and $h \perp g$. By 3.2.1.2, there is an A in h such that $A \neq B$. By 3.3.8, there is an $l \in \mathcal{L}_\alpha$ such that $A \in l$ and $l \perp h$. By 3.2.1.2, there exists $Q \in l$ such that $Q \neq A$.

Now if $Q \in h$, then $A, Q \in l, h$ implies that $Q = A$ or $l = h$, so $Q \notin h$. Suppose that $Q \in g$. Let $Q \in m$, $m \perp l$, so that $\sigma_Q = \sigma_l \sigma_m$. Because $\sigma_B = \sigma_g \sigma_h$, $\sigma_A = \sigma_h \sigma_l$, and $BAQ = E \in \mathfrak{X}_\alpha$, then

$$\sigma_E = \sigma_B \sigma_A \sigma_Q = \sigma_g \sigma_h \sigma_h \sigma_l \sigma_l \sigma_m = \sigma_g \sigma_m \text{ and } g \perp m.$$

Because $Q \in g, m$ and $g \perp m$ then $\sigma_Q = \sigma_g \sigma_m = \sigma_l \sigma_m$ and $\sigma_g = \sigma_l$ or $g = l$. This implies $A, B \in l, m$. Thus $A = B$ or $l = m$, a contradiction. ■

Axiom 6: *There exist $A, B; A \neq B$, such that $A, B \not| g$ for any $g \in \mathcal{G}$.*

Proof: This follows from our Axiom 13. ■

Axiom 7: *For each P and each A, B, C such that $A, B, C | g$, there is a v in $\mathcal{G}$ such that $P, A | v$, or $P, B | v$, or $P, C | v$.*

Proof: By our Axiom 13, there exist two isotropic lines in $\mathfrak{X}_\alpha$ through P, say g_{PQ} and g_{PR}. If no such v in $\mathcal{L}_\alpha$ satisfies the above, then two of the three points must lie on one of the isotropic lines by our Axiom 13. But this implies that $g_{PQ} = g$ or $g_{PR} = g$; that is, g is isotropic; a contradiction. ■

3.4. Consequences of Section 3.3

3.4.1. *For every A and B, there exist $\alpha \in \mathcal{M}$ such that $A, B | \alpha$. (Every line lies in some Minkowskian plane.)*

Proof: By Axiom 6, there exist $\alpha \in \mathcal{P}$ such that $A, B | \alpha$. If $\alpha \in \mathcal{M}$, the result follows. So suppose that $\alpha \in \mathcal{G}$. By Axiom 12, there exist $\beta \in \mathcal{G}$ such that $A, B | \beta$ and $\beta \perp \alpha$. Then $A, B | \alpha\beta = \gamma$, and $\gamma \in \mathcal{M}$ by the definition of $\mathcal{M}$. ■

3.4.2. *For every A and B, there exists M such that $A^M = B$; that is, every two points has a midpoint and by* 3.1.6.14, *the midpoint is unique.*

Proof: By 3.4.1 above, there exists $\alpha \in \mathcal{M}$ such that $A, B | \alpha$. From Section 3.3, there exists $M | \alpha$ such that $A^M = B$. ■

3.4.3. *If $AA' = BB'$ then A and A' are joinable precisely when B and B' are.*

Proof: Suppose that A and A' are joinable and let $A, A' | \alpha, \alpha'$ with $\alpha \perp \alpha'$. By 3.4.2, there exists an M such that $A^M = B$. Then $B = A^M | \alpha^M, \alpha'^M$ and from 3.1.2.1 and 3.1.6.13 it follows that $\alpha^M \perp \alpha'^M$, $\alpha \parallel \alpha^M$, and $\alpha' \parallel \alpha'^M$. By 3.1.6.11,

$$AA' = BB'; \quad A, A' | \alpha, \alpha'; \quad B | \alpha^M, \alpha'^M; \quad \alpha \parallel \alpha^M; \quad \text{and} \quad \alpha' \parallel \alpha'^M.$$

Thus, $B' | \alpha^M, \alpha'^M$ and, B and B' are joinable. ■

3.4.4. *If $\alpha \perp \alpha'$; $\alpha \parallel \beta$; $\alpha' \parallel \beta'$; and $[\alpha, \alpha'] \parallel [\beta, \beta']$, then $\beta \perp \beta'$.*

Proof: Because $[\alpha, \alpha'] \parallel [\beta, \beta']$, then there exist A, A'; $A, A' | \alpha, \alpha'$ and there exist B, B'; $B, B' | \beta\beta'$, such that $AA' = BB'$. Then for $A^M = B$, as in the proof of 3.4.3, $[\beta, \beta'] = g_{BB'} = [\alpha^M, \alpha'^M] \parallel [\alpha, \alpha']$. That is, $B | \alpha^M, \alpha'^M, \beta, \beta'$; with $\beta, \alpha^M \parallel \alpha$, and $\beta', \alpha'^M \parallel \alpha'$. Thus, $\beta = \alpha^M$ and $\beta' = \alpha'^M$ by 3.1.6.12. Hence, $\beta \perp \beta'$ because $\alpha^M \perp \alpha'^M$. ■

3.5. Construction of the Field

The basic construction. For the construction of the field we will follow the path of Lingenberg [20]. Throughout this section let $\alpha \in \mathcal{M}$ and $O | \alpha$ be fixed. Define the sets:

$$O_\alpha \equiv \{g \in \mathcal{L}_\alpha : O \in g\} \quad \text{and} \quad \mathcal{D}_\alpha(O) \equiv \{\sigma_g \sigma_h : g, h \in O_\alpha\}.$$

Proposition 3.5.1 *The set $\mathcal{D}_\alpha(O)$, acting on the points of $\mathfrak{X}_\alpha$, is an abelian group.*

Proof: By 3.1.7.2 we may write $O = \alpha\beta = \gamma\eta\beta$ with $\alpha = \gamma\eta$; $\gamma, \eta, \beta \in \mathcal{G}$; and γ, η, and β mutually orthogonal. Thus, $g = [\alpha, \gamma, \eta] \in \mathcal{L}_\alpha$; $O \in g$, and $O_\alpha \neq \varnothing$. Because each

$\sigma_g \in \mathcal{C}_\alpha$ is involutory then $1_{\mathfrak{X}_\alpha} \in \mathcal{D}_\alpha(O)$. Now let $\sigma_a, \sigma_b, \sigma_c, \sigma_d \in \mathcal{D}_\alpha(O)$ where $a = [\alpha, \alpha', \alpha\alpha']$, $b = [\alpha, \beta', \alpha\beta']$, $c = [\alpha, \gamma', \alpha\gamma']$, and $d = [\alpha, \delta, \alpha\delta]$ are nonisotropic. Now $\delta\gamma'\beta' = \varepsilon \perp \alpha$ with $O|\varepsilon$ by Axiom 9 and $f = [\alpha, \varepsilon, \alpha\varepsilon] \in O_\alpha$. Thus, for $A|\alpha$,

$$\sigma_a\sigma_b\sigma_c\sigma_d(A) = A^{\delta\gamma'\beta'\alpha'} = A^{\varepsilon\alpha'} = \sigma_a\sigma_f(A).$$

Hence, $\sigma_a\sigma_b\sigma_c\sigma_d = \sigma_a\sigma_f \in \mathcal{D}_\alpha(O)$. Because each σ_l is involutory for $l \in \mathcal{L}_\alpha$, $\sigma_l^{-1} = \sigma_l$ and $(\sigma_a\sigma_b)^{-1} = \sigma_b\sigma_a \in \mathcal{D}_\alpha(O)$. From Axiom 9 and the calculation above, the product of any three of δ, γ', β', and α' is an involution.Thus, $A^{\delta\gamma'\beta'\alpha'} = A^{\beta'\gamma'\delta\alpha'} = A^{\beta'\alpha'\delta\gamma'}$ and $\sigma_a\sigma_b\sigma_c\sigma_d = \sigma_c\sigma_d\sigma_a\sigma_b$.That is, $\mathcal{D}_\alpha(O)$ is abelian. Clearly, $\mathcal{D}_\alpha(O)$ is associative so that $\mathcal{D}_\alpha(O)$ is indeed an abelian group. ∎

Lemma 3.5.2. *Let g be an isotropic line in $\mathfrak{X}_\alpha$ with $O \in g$. Then for every $\sigma_l\sigma_h \in \mathcal{D}_\alpha(O)$, $\sigma_l\sigma_h(g) = g$.*

Proof: Let $l = [\alpha, \beta, \alpha\beta]$ and $h = [\alpha, \gamma, \alpha\gamma]$ be lines in $\mathcal{L}_\alpha$ with $O \in l, h$. By 3.2.3.12, if $O|\beta, \gamma$ with $\beta, \gamma \perp \alpha$ then $g^{\beta\gamma} \parallel g$. But $O^{\beta\gamma} = O$ so that $\sigma_h\sigma_l(O) = O^{\beta\gamma} = O$. Thus, $g^{\beta\gamma} = g$ by 3.2.1.10. Hence, $\sigma_h\sigma_l(O) = g$, for all $\sigma_h\sigma_l \in \mathcal{D}_\alpha(O)$. ∎

Lemma 3.5.3. *Let g be an isotropic line in $\mathfrak{X}_\alpha$ through O. Then for every $A, E \in g$, $E \neq O$, $A \neq O$, there is a unique $\sigma_l\sigma_h \in \mathcal{D}_\alpha(O)$ such that $\sigma_l\sigma_h(E) = A$.*

Proof: By 3.2.3.11, there exist $\gamma, \delta \perp \alpha$ with $O|\gamma, \delta$ such that $E^{\gamma\delta} = A$. Take $l = [\alpha, \gamma, \alpha\gamma]$ and $h = [\alpha, \delta, \alpha\delta]$. Note that if $E = A$ then $E^{\gamma\delta} = E$ implies that $E^\gamma = E^\delta$ or $E^l = E^h$, which implies that $l = h$ [30]. To show uniqueness, suppose that $\sigma_a\sigma_h, \sigma_k\sigma_l \in \mathcal{D}_\alpha(O)$, where $l = [\alpha, \delta, \alpha\delta]$, $k = [\alpha, \varepsilon, \alpha\varepsilon]$, $g = [\alpha, \gamma, \alpha\gamma]$, and $h = [\alpha, \beta, \alpha\beta]$ are lines in $\mathfrak{X}_\alpha(O)$ and $E \neq E^{ha} = E^{lk}$. Then $E^{hakl} = E$ and $E^{\beta\gamma\varepsilon\delta} = E$. By Axiom 9, $\beta\gamma\varepsilon = \lambda$ with $O|\lambda$, $\lambda \perp \alpha$. Thus, $m = [\alpha, \lambda, \alpha\lambda] \in O_\alpha; \sigma_k\sigma_a\sigma_h = \sigma_m$, and $E = E^{ml}$. Let $E|\alpha', \delta'$ with $\alpha' \hat{I} \lambda$, $\delta' \hat{I} \delta$, and put $M = \alpha'\lambda$ and $N = \delta'\delta$. Now $E|\alpha', \alpha, \delta'$ with $\alpha' \hat{I} \lambda$, $\alpha \perp \lambda$, $\delta' \perp \delta$, and $\alpha \perp \delta$, so that $\alpha', \delta' \perp \alpha$ by 3.1.6.18. It follows that $M^\alpha = \alpha'^\alpha\lambda^\alpha = \alpha'\lambda = M$ and $N^\alpha = \delta'^\alpha\delta^\alpha = N$. So that $M|\alpha, \lambda$; $N|\alpha, \delta$; and $m = g_{OM}$ and $l = g_{ON}$. Then because $E^{ml} = E$, we have $E^{\lambda\delta} = E$; $E^\lambda = E^\delta$ and

$E^M = E^{\alpha'\lambda} = E^{\delta'\delta} = E^N$ and $M = N$ by 3.1.6.14. So $m = g_{OM} = g_{ON} = l$ and $\sigma_l = \sigma_m$. Therefore $\sigma_a\sigma_h = \sigma_k\sigma_l$. ■

Let us denote this unique map by δ_A. So for all $A \in g_{OE}$, $A \neq O$,

(i) $\delta_A(O) = O$

(ii) $\delta_A(E) = A$

(iii) $\delta_A \in \mathcal{D}_\alpha(O)$.

Translations. For every pair A,B of distinct points we can define a translation $T_{AB} : \mathfrak{X} \mapsto \mathfrak{X}$ given by $T_{AB}(A) = AAB = B$. We now restrict our attention to a set of translations defined on $\mathfrak{X}_\alpha$ and we note that if $A,B,C|\alpha$, then $D = ABC|\alpha$ by 3.1.4.6. Thus, $T_{AB} : \mathfrak{X}_\alpha \mapsto \mathfrak{X}_\alpha$ is a well-defined map for all $A,B|\alpha$. Let $\mathcal{K} \equiv g_{OE}$ be an isotropic line in $\mathfrak{X}_\alpha$ and define $\mathcal{T}_\alpha \equiv \{T_{OA} : A \in \mathcal{K}\}$.

Theorem 3.5.4. *The set $\mathcal{T}_\alpha$ is an abelian group.*

Proof: Let $T_{OA}, T_{OB} \in \mathcal{T}_\alpha$. Because $C = BOA = AOB \in \mathcal{K}$, for $X|\alpha$, we calculate $(T_{OA} \circ T_{OB})(X) = T_{OA}(XOB) = XOBOA = XOC = T_{OC}(X)$. Thus, $T_{OC} = T_{OA} \circ T_{OB} \in \mathcal{T}_\alpha$. Also, $T_{OO}(X) = XOO = X$. Hence, $T_{OO} \in \mathcal{T}_\alpha$ is the identity $1_{\mathfrak{X}_\alpha}$ on $\mathfrak{X}_\alpha$. To find T_{OA}^{-1}, we compute

$$(T_{OA} \circ T_{OA^O})(X) = XOA^OOA = XOOAOOA = X = XOAOA^O = (T_{OA^O} \circ T_{OA})(X).$$

Hence, $T_{OA}^{-1} = T_{OA^O}$, $A^O = OAO \in \mathcal{K}$, and $T_{OA}^{-1} \in \mathcal{T}_\alpha$. Clearly, $T_{OA} \circ (T_{OB} \circ T_{OC}) = (T_{OA} \circ T_{OB}) \circ T_{OC}$ for $A,B,C \in \mathcal{K}$. Therefore the action of $\mathcal{T}_\alpha$ is associative and $\mathcal{T}_\alpha$ is a group. Because ABC is a point and hence, an involution for all points A,B,C, then for $A,B \in \mathcal{K}$ and $X|\alpha$,

$$(T_{OA} \circ T_{OB})(X) = XOBOA = XOAOB = (T_{OB} \circ T_{OA})(X).$$

Therefore, $\mathcal{T}_\alpha$ is abelian. ■

Lemma 3.5.5. *For all $T_{OA} \in \mathcal{T}_\alpha$, $T_{OA}(\mathcal{K}) = \mathcal{K}$.*

Proof: Because $OAB \in \mathcal{K}$ for $A,B \in \mathcal{K}$ then $T_{OA}(\mathcal{K}) \subset \mathcal{K}$ for all $T_{OA} \in \mathcal{T}_\alpha$. Now let

$C \in \mathcal{K}$ and $A \in \mathcal{K}$. Then $OCA = D \in \mathcal{K}$ so that $C = ODA$ and $T_{OA}(D) = C$. Hence, each $T_{OA} \in \mathcal{T}_\alpha$ maps $\mathcal{K}$ onto $\mathcal{K}$. ■

Lemma 3.5.6. *Let $T_{OA} \in \mathcal{T}_\alpha$ and $g \subset \mathfrak{X}_\alpha$. Then $T_{OA}(g) = g$ if and only if g* is parallel to $\mathcal{K}$.

Proof: Let $g_{HF} = g$ be a line in $\mathfrak{X}_\alpha$ such that $T_{OA}(g_{HF}) = g_{HF}$. Then $T_{OA}(H) = HOA \in g_{HF}$ and $g_{HF} \parallel \mathcal{K}$ by 3.2.1.9. Conversely, suppose that $g_{HF} \parallel \mathcal{K}$. Then again by 3.2.1.9 it follows that for $B \in g_{HF}$,

$$T_{OA}(B) = BOA \in g_{HF} \text{ and } T_{OA}(g) = g. \quad ■$$

Lemma 3.5.7. *If $A, B \in h$, $h \subset \mathfrak{X}_\alpha$, and $h \parallel \mathcal{K}$, then there exists $T_{OC} \in \mathcal{T}_\alpha$ such that $T_{OC}(A) = B$.*

Proof: By 3.2.1.9 we have $C = OAB \in \mathcal{K}$ and it follows that $B = AOC = T_{OC}(A)$. ■

Lemma 3.5.8. *For each $A \in \mathcal{K}$, there is a unique $T_{OA} \in \mathcal{T}_\alpha$ such that $T_{OA}(O) = A$.*

Proof: Clearly, $T_{OA}(O) = OOA = A$. So suppose that $T_{OC}(O) = A$. Then $OOC = C = A$. ■

We denote this unique translation mapping of O into A by T_A.

Lemma 3.5.9. *If $\sigma \in \mathcal{D}_\alpha(O)$ and $T_A \in \mathcal{T}_\alpha$, then $\sigma T_A \sigma^{-1} = T_{\sigma(A)}$.*

Proof: Let $\sigma = \sigma_g\sigma_h$ where $g = [\alpha, \beta, \alpha\beta]$, $h = [\alpha\gamma, \alpha\gamma] \in O_\alpha$. Then for any $X|\alpha$,

$$(\sigma_g\sigma_h T_A \sigma_h \sigma_g)(X) = (X^{\beta\gamma}OA)^{\gamma\beta} = (X^{\beta\gamma})^{\gamma\beta}O^{\gamma\beta}A^{\gamma\beta} = XOA^{\gamma\beta} = T_{\sigma_g\sigma_h(A)}(X). \ ■$$

By 3.2.3.8 there exist precisely two isotropic lines in $\mathfrak{X}_\alpha$ through $O : \mathcal{K} \equiv g_{OE}$ and $\mathcal{K}' \equiv g_{OF}$. We define multiplication and addition on the points of $\mathcal{K}$ so that the points of $\mathcal{K}$ form a field. For $A, B \in \mathcal{K}$, define:

$A + B \equiv (T_B \circ T_A)(O)$

$A \cdot B \equiv (\delta_B \circ \delta_A)(E)$, where $A, B \neq O$ and $E \in \mathcal{K}$ is the multiplicative identity.

$A \cdot O \equiv O \cdot A \equiv O$.

Theorem 3.5.10. *For every $A, B \in \mathcal{K}$, $T_{A+B} = T_A \circ T_B$.*

Proof: For $X|\alpha$, $T_{A+B}(X) = T_{AOB}(X) = XOAOB = XOBOA = (T_A \circ T_B)(X)$. ■

Theorem 3.5.11. *For all $A,B \neq O$ in $\mathcal{K}$, $\delta_{A \cdot B} = \delta_A \circ \delta_B$.*

Proof: Let $\delta_A = \sigma_a\sigma_{a'}$ and $\delta_B = \sigma_b\sigma_{b'}$. Then $A \cdot B \equiv \delta_B\delta_A(E) = E^{a'ab'b}$. Put $\sigma_c = \sigma_{b'}\sigma_a\sigma_{a'}$, then, $\sigma_b\sigma_c \in \mathcal{D}_\alpha(O)$ and $\sigma_b\sigma_c(E) = E^{cb} = E^{a'ab'b} = A \cdot B$. Hence, by 3.5.1.3, $\delta_{A \cdot B} = \sigma_b\sigma_c = \sigma_b\sigma_{b'}\sigma_a\sigma_{a'} = \sigma_a\sigma_{b'}\sigma_b\sigma_{a'} = \sigma_a\sigma_{a'}\sigma_b\sigma_{b'} = \delta_A\delta_B$. ■

Hence, $(\mathcal{K},+)$ is a group isomorphic to $\mathcal{T}_\alpha$ and $(\mathcal{K}\backslash\{O\},\cdot)$ is a group isomorphic to $\mathcal{D}_\alpha(O)$. It remains to show that the distributive laws hold.

Theorem 3.5.12. *Let $A,B,C \in \mathcal{K}$, then $(A+B) \cdot C = A \cdot C + B \cdot C$.*

Proof: If $C = O$, then

$$(A+B) \cdot C = O = O \cdot O + O \cdot O = A \cdot C + B \cdot C.$$

If $C \neq O$, then we compute

$$\begin{aligned}(A+B) \cdot C = \delta_C(A+B) = \delta_C T_{A+B}(O) = \delta_C T_{A+B}\delta_C^{-1}(O) = \delta_C T_A T_B \delta_C^{-1}(O) = \delta_C T_A \delta_C^{-1}\delta_C T_B \delta \\ = T_{\delta_C(A)}T_{\delta_C(B)}(O) = T_{A \cdot C}T_{B \cdot C}(O) = T_{A \cdot C + B \cdot C}(O) = A \cdot C + B \cdot C.\end{aligned}$$

Because multiplication is commutative,

$C \cdot (A+B) = (A+B) \cdot C = A \cdot C + B \cdot C = C \cdot A + C \cdot B$. Hence, $(\mathcal{K},+,\cdot)$ is a field. ■

Ordering the field and obtaining $\mathbb{R}$. To order the field $\mathcal{K}$ we make use of the following [21]. Let $\mathbb{F}$ be a field and $A_1,\ldots,A_n \in \mathbb{F}$. If $A_1,\ldots,A_n \neq 0$ implies that $\sum_{k=1}^n A_k^2 \neq 0$, then $\mathbb{F}$ is called *formally real.*

Theorem 3.5.13. *(Artin-Schreier) Every formally real field can be ordered.* ■

To make the field $\mathcal{K}$ formally real the following axiom is posited.

Axiom F. *(formally real axiom). Let $O,E|\alpha$, $\alpha \in \mathcal{M}$ with O and E unjoinable. Let $\lambda,\delta,\eta \in \mathcal{P}$. If $O|\lambda,\delta,\eta$ and $\lambda,\delta,\eta \perp \alpha$ then there is a $\gamma \in \mathcal{P}$ such that*

$$O|\gamma,\ \gamma \perp \alpha,\ \textit{and}\ E^{\lambda\delta\lambda\delta}OE^{\lambda\eta\lambda\eta} = E^{\lambda\gamma\lambda\gamma}.$$

Theorem 3.5.14. *If Axiom F holds on $\mathcal{K}$, then $\mathcal{K}$ is formally real.*

Proof: Let $t = [\alpha,\lambda,\alpha\lambda] \in O_\alpha$ and let $\sigma_g\sigma_h \in \mathcal{D}_\alpha(O)$. Then from Proposition 3.5.1,

$\sigma_g\sigma_h\sigma_t = \sigma_l$ for some $l \in O_\alpha$ and $\sigma_g\sigma_h = \sigma_l\sigma_t$. Clearly, if $l' \in O_\alpha$ with $\sigma_{l'}\sigma_t = \sigma_l\sigma_t$ then we have $\sigma_{l'} = \sigma_l$ and $l' = l$. Hence, for all $\sigma_g\sigma_h \in \mathcal{D}_\alpha(O)$, there is a unique $l \in O_\alpha$, such that $\sigma_g\sigma_h = \sigma_l\sigma_t$. So if $O \neq A \in \mathcal{K}$, then we may uniquely write δ_A in the form $\sigma_a\sigma_t$; that is,for every $O \neq A \in \mathcal{K}$, there is a unique a, $a = [\alpha,\alpha',\alpha\alpha'] \in O_\alpha$ such that

$$A = \sigma_a\sigma_t(E) = E^{ta} = E^{\lambda\alpha'}.$$

Suppose that $O \neq A = E^{\lambda\alpha'}$. Then $A^2 = E^{\lambda\alpha'\lambda\alpha'} = O$ implies that $E = O^{\alpha'\lambda\alpha'\lambda} = O$, a contradiction. Hence, if $O \neq A \in \mathcal{K}$ then $A^2 \neq O$. Now suppose that $A,B \in \mathcal{K}$ and $A,B \neq O$. Let $a = [\alpha,\alpha',\alpha\alpha']$, $b = [\alpha,\beta,\alpha\beta] \in O_\alpha$ such that $A = E^{\lambda\alpha'}$ and $B = E^{\lambda\beta}$. Then by Axiom F there exists a $\gamma \in \mathcal{P}$ such that $O|\gamma \perp \alpha$ and

$$A^2 + B^2 = E^{\lambda\alpha'\lambda\alpha'}OE^{\lambda\beta\lambda\beta} = E^{\lambda\gamma\lambda\gamma}. \tag{3.5.1}$$

Since $O|\gamma \perp \alpha$ then $c = [\alpha,\gamma,\alpha\gamma] \in O_\alpha, \sigma_c\sigma_t \in \mathcal{D}_\alpha(O)$, and

$$C = \sigma_c\sigma_t(E) = E^{tc} = E^{\lambda\gamma} \in \mathcal{K}.$$

So equation (3.5.1) reads $A^2 + B^2 = C^2$. If $C^2 = E^{\lambda\gamma\lambda\gamma} = O$, then $E = O^{\gamma\lambda\gamma\lambda} = O$. Since $E \neq O$, it follows that if $A_1,\ldots,A_n \in \mathcal{K}$ are all nonzero, then $\sum_{k=1}^{n} A_k^2 \neq O$ and $\mathcal{K}$ is formally real. ■

To finally obtain a field isomorphic to the real numbers, $\mathbb{R}$, we add the least upper bound property to our axiom system.

Axiom L. *If $\varnothing \neq \mathcal{A} \subset \mathcal{K}$ and $\mathcal{A}$ is bounded above, then there exists an $A \in \mathcal{K}$ such that $A \geq X$, for all $X \in \mathcal{A}$, and if $B \in \mathcal{K}$ with $B \geq X$ for all $X \in \mathcal{A}$ then $A \leq B$.*

The only ordered field up to isomorphism with the least upper bound property is the real number field, $\mathbb{R}$.

Theorem 3.5.15. *The field $\mathcal{K}$ constructed above, along with Axiom F and Axiom L, is isomorphic to the real number field, $\mathbb{R}$.*

In the next section an affine vector space is constructed from products of pairs of points. The scalar multiplication is obtained by adapting and extending the definition of multiplication of elements of $\mathcal{K}$.

3.6. Dilations and the Construction of $(\mathfrak{X}, \mathcal{V}, \mathcal{K})$

The additive group $\mathcal{V}$ and dilations. First we construct a vector space $\mathcal{V}$ over the field $\mathcal{K}$. Let $\mathcal{V} = \{OX : X \in \mathfrak{X}\}$. First note that the product, AB, of any two points $A, B \in \mathfrak{X}$, is in $\mathcal{V}$ because $AB = O(OAB) = OOAB$. We view the elements $OX \in \mathcal{V}$ as directed line segments with initial point O and terminal point X on the line g_{OX}. We define an addition on $\mathcal{V}$ by setting $OX + OY \equiv OXOY$. The product of three points is a point, so $XOY = Z \in \mathfrak{X}$, $OXOY = OZ \in \mathcal{V}$.

Theorem 3.6.1. $(\mathcal{V}, +)$ *is an abelian group.*

Proof: Let $X, Y, Z \in \mathfrak{X}$ be distinct. Then, $OX + OY = OXOY = OYOX = OY + OX$, and addition is abelian. The zero vector.is $1_{\mathfrak{G}}$ since $1_{\mathfrak{G}} = OO \in \mathcal{V}$ and $1_{\mathfrak{G}} + OX = 1_{\mathfrak{G}}OX = OX = OX + 1_{\mathfrak{G}}$. To complete the proof we calculate

$$OX + OX^O = OXOX^O = OXOOXO = OXXO = OO = 1_{\mathfrak{G}}, \text{ so } -OX = OX^O.$$
$$(OX + OY) + OZ = (OXOY)OZ = OX(OYOZ) = OX + (OY + OZ).$$

Hence, $(\mathcal{V}, +)$ is an abelian group. ■

We still need to define a scalar multiplication of $\mathcal{K}$ on $\mathcal{V}$. To do this note that in an affine space the group of dilations with fixed point C is isomorphic to the multiplicative group of the field. Noting this, we geometrically construct such a group of mappings and use these mappings to define our scalar multiplication.

A *dilation* of $\mathfrak{X}$ is a mapping $\delta : \mathfrak{X} \mapsto \mathfrak{X}$ which is bijective and which maps every line of $\mathfrak{X}$ onto a parallel line.[23, p.37]

Theorem 3.6.2. [23, p.42] *A dilation δ is completely determined by the images of two points.*

Proof: Let $\delta : \mathfrak{X} \mapsto \mathfrak{X}$ be a dilation and assume that $\delta(X) = X'$ and $\delta(Y) = Y'$ of two points $X, Y \in \mathfrak{X}$ are known. We must show that the image of any $Z \in \mathfrak{X}$ is known. Suppose that $Z \notin g_{XY}$. Then clearly $Z \neq X$ and $Z \neq Y$ and we consider the two lines g_{XZ} and g_{YZ}. Observe that $Z \in g_{XZ} \cap g_{YZ}$. If these lines had a point in common besides Z, they would be equal and then $g_{XZ} = g_{YZ} = g_{XY}$. But this is impossible because $Z \notin g_{XY}$. Therefore, $\{Z\} = g_{XZ} \cap g_{YZ}$. Since δ is bijective, from set-theoretic reasons alone, $\delta(g_{XZ} \cap g_{YZ}) = \delta(g_{XZ}) \cap \delta(g_{YZ})$; or equivalently, $\{\delta(Z)\} = \delta(g_{XZ}) \cap \delta(g_{YZ})$. In other words, the lines $\delta(g_{XZ})$ and $\delta(g_{YZ})$ have precisely one point in common, namely, the point $\delta(Z)$ for which we are looking. The line $\delta(g_{XZ})$ is completely known because it is the unique line which passes through X' and is parallel to g_{XZ}. Similarly, the line $\delta(g_{YZ})$ is the unique line which passes through the point Y' and is parallel to g_{YZ}. because $\delta(Z)$ is the unique point of intersection of $\delta(g_{XZ})$ and $\delta(g_{YZ})$ (3.2.1.1), the point $\delta(Z)$ is completely determined by X' and Y'.

Conversely, assume that $Z \in g_{XY}$. If Z is X or Y, we are given $\delta(Z)$, so assume that $Z \neq X$ and $Z \neq Y$. By 3.4.1, there is an $\alpha \in \mathcal{M}$ such that $X, Y | \alpha$ and there exists a $P | \alpha$ such that $P \notin g_{XY}$. Then $Z \notin g_{XP}$ and from the previous paragraph, $\delta(P)$ is known. Hence, using the line g_{XP} instead of the line g_{XY}, we conclude from the earlier proof that $\delta(Z)$ is known. ■

To define a scalar multiplication, we fix a timelike line t and use it to geometrically define dilations. To aid in the construction, the following facts are used to add the appropriate axioms.

In a three-dimensional or four-dimensional Minkowski space, if t is any timelike line through a point O and g is any other line through O, then there is a unique Lorentz plane containing the two lines. Two distinct isotropic lines intersecting in a point in Minkowski space determine a unique Lorentz plane. Desargue's axiom, D, holds in any affine space of dimension $d \geq 3$.

Axiom T. *If* $O \in t, g$ *where* t *is timelike or* t *and* g *are both isotropic then is a unique* $\alpha \in \mathcal{M}$ *such that* $g, t \subset \mathfrak{X}_\alpha$.

Axiom D. *Let* g_1, g_2, *and* g_3 *be any three distinct lines, not necessarily coplanar, which intersect in a point* O. *Let* $P_1, Q_1 \in g_1$; $P_2, Q_2 \in g_2$; *and* $P_3, Q_3 \in g_3$. *If* $g_{P_1P_3} \parallel g_{Q_1Q_3}$ *and* $g_{P_2P_3} \parallel g_{Q_2Q_3}$ *then* $g_{P_1P_2} \parallel g_{Q_1Q_2}$.

Axiom R. *Let* $O \in g_1, g_2$; $P_1, Q_1 \in g_1$; *and* $P_2, Q_2 \in g_2$. *If* $g_{P_1P_2} \parallel g_{Q_1Q_2}$ *then* $g_{O,P_1OP_2} = g_{O,Q_1OQ_2}$.

Axiom T refers to the first statements. Axiom T is used to put an isomorphic copy of the field on every isotropic line through O. A scalar multiplication is then defined in a manner similar to the definition of multiplication for the field elements. Axiom D is Desargue's axiom, the "dilation" axiom. Axiom D ensures well-defined dilations with the standard properies of such maps. Axiom R is used to distribute a scalar over the sum of two vectors. The dilations are constructed next.

Let $\mathcal{K} \subset \mathfrak{X}_\alpha$, $\alpha \in \mathcal{M}$. By 3.1.7.2, there exist $\alpha_1, \alpha_2 \in \mathcal{G}$ such that $O|\alpha_1, \alpha_2$ and $\alpha = \alpha_1\alpha_2$. Because $O|\alpha$, then $O\alpha = \beta \in \mathcal{G}$ and $O = \alpha\beta = \alpha_1\alpha_2\beta$ with $\alpha_1, \alpha_2 \perp \beta$ by 3.1.6.18. Let $\gamma = \alpha_1\beta \in \mathcal{M}$. Then $\gamma\alpha = \beta\alpha_1\alpha_1\alpha_2 = \beta\alpha_2 \in \mathcal{M}$, so that $t = [\alpha, \gamma, \alpha\gamma] \subset \mathfrak{X}_\alpha$ is timelike and $O \in t$ as $O|\alpha, \alpha_1, \alpha_2, \beta$ implies that $O|\alpha, \gamma, \alpha\gamma$. So let $t = [\alpha, \gamma, \delta] \in O_\alpha$ be any timelike line through O and $\mathcal{K}' \subset \mathfrak{X}_\alpha$ the other isotropic line through O. Then for each $A_t \in t$, there is a unique $A \in \mathcal{K}$ and there is a unique $B \in \mathcal{K}'$ such that $A_t = AOB$. Because $A_t \in t$, then $B = A^t$. (From this point on denote a line reflection $\sigma_g(X)$ by X^g. Thus, X^{ghr} means $\sigma_r\sigma_h\sigma_g(X) = \sigma_g\sigma_h\sigma_r(X)$.).

For each $A_t \in t$, there is a unique $A \in \mathcal{K}$ such that $A_t = AOA^t$. Similarly, for each A in $\mathcal{K}$, there is a unique A_t in t such that $A_t = AOA^t$. Thus, there is a one-to-one correspondence between the points of $\mathcal{K}$, the field, and the points of t.

Fix $E_t = EOE^t$ as the unit point on t. For each $O \neq A \in \mathcal{K}$, we use t to construct a dilation $\overline{\delta}_A$ of $\mathfrak{X}$ in the following way. Let $X \in \mathfrak{X}$.

If $X \notin t$, then by Axiom T, there is a unique $\eta \in \mathcal{M}$ such that $g_{OX}, t \subset \mathfrak{X}_\eta$. From Sections 3.5 and 3.6, $\mathfrak{X}_\eta$ is an affine plane and our definition of parallel lines in $\mathfrak{X}_\eta$ is equivalent to the affine definition. So, there is a unique line $h \subset \mathfrak{X}_\eta$ such that $A_t \in h$ and $h \parallel g_{XE_t}$. Because

$$g_{OX} \cap t = \{O\} \neq \varnothing,\ h \cap t = \{A_t\} \neq \varnothing,\ \text{and } h \parallel g_{E_tX},$$

then $h \cap g_{OX} = \{B\} \neq \varnothing$. In this case, set $\bar{\delta}_A(X) = B$.

If $X \in t$ and $X \neq O$, then because $t, \mathcal{K} \subset \mathfrak{X}_\alpha$ with $\alpha \in \mathcal{M}$, there exists a unique $g \subset \mathfrak{X}_\alpha$ such that $A \in g$ and $g \parallel g_{XE}$. Because $g_{XE} \cap t = \{E\} \neq \varnothing$ then $g \cap t = \{B\} \neq \varnothing$. Set $\bar{\delta}_A(X) = B$.

If $X = O$, set $\bar{\delta}_A(X) = O$. Note that $\bar{\delta}_A(X) \in g_{OX}$ by construction. It is clear that $\bar{\delta}_A : \mathfrak{X} \mapsto \mathfrak{X}$ is one-to-one. We find it useful to make some observations.

Let $O \neq A \in \mathcal{K}$ and recall that $E \in \mathcal{K}$ is the multiplicative unit point; $E \cdot A = A$, for $A \in \mathcal{K}$. Put $A_t = AOA^t$.

Lemma 3.6.3. *For the map $\bar{\delta}_A$ defined above, the following are true:*

1 $\bar{\delta}_A(E) = A$.

2 $\bar{\delta}_A(E_t) = A_t$.

3 $\bar{\delta}_A$ *is onto.*

Proof: 1. $E_t = EOE^t$ and $A_t = AOA^t$ imply that $EE_t = OE^t$ and $AA_t = OA^t$. If $EE_t = OE^t = 1_{\mathfrak{G}}$ then $O = E^t$ and $O = O^t = E$. Similarly, because $A \neq O$ then $AA_t = OA^t \neq 1_{\mathfrak{G}}$. By 3.2.2.9, $A^t = A^\gamma \in \mathcal{K}'$ as $O, A \in \mathcal{K}$, $\mathcal{K}$ is isotropic, and $O|\gamma \perp \alpha$. Similarly, $E^t \in \mathcal{K}'$. Thus by 3.2.1.6 we have $g_{EE_t} \parallel g_{OE^t} = \mathcal{K}' \parallel g_{AA^t}$ and $g_{EE_t} \parallel g_{AA_t}$ by 3.2.1.7. By 3.2.1.8, $h = g_{AA_t}$ and $h \cap g_{OE} = g_{AA_t} \cap \mathcal{K} = \{A\}$. Hence $\bar{\delta}_A(E) = A$.

2 $\bar{\delta}_A(E_t) = A_t$. Again, $g_{\bar{\delta}_A(E)A_t} = g_{AA_t} \parallel g_{EE_t}$ and $g_{AA_t} \cap t = \{A_t\}$.

3 Let $P \in \mathfrak{X}$ and $P \neq O$. If $P \notin t$ then by 3.2.1.8, there is a unique line g such that $E_t \in g$, $g \parallel g_{A_tP}$, and $g \cap g_{OP} = \{Q\}$, for some $Q \in \mathfrak{X}$. Then it follows that $\bar{\delta}_A(Q) = P$. If $P \in t$ then $\bar{\delta}_A(Q) = P$, where $\{Q\} = g \cap t$, $E \in g$, and $g \parallel g_{AP}$. ■

Lemma 3.6.4. *If $C \neq D$, then $g_{CD} \parallel g_{\bar{\delta}_A(C)\bar{\delta}_A(D)}$; hence, for all $O \neq A \in \mathcal{K}$, $\bar{\delta}_A$ is a dilation of $\mathfrak{X}$.*

Proof: Let $g_1 = t$, $g_2 = g_{OC}$, $g_3 = g_{OD}$, $P_1 = E_t$, $Q_1 = A_2 \in g_1$, $P_2 = C$, $Q_2 = \bar{\delta}_A(C)$, $Q_2 \in g_2$, $P_3 = D$, and $Q_3 = \bar{\delta}_A(D)$. Then $g_{CE_t} \parallel g_{A_t\bar{\delta}_A(C)}$ and $g_{DE_t} \parallel g_{A_t\bar{\delta}_A(D)}$. Thus, by Axiom D, $g_{CD} \parallel g_{\bar{\delta}_A(C)\bar{\delta}_A(D)}$. ■

Consider now the plane $\mathfrak{X}_\alpha$. Recall that by 3.5.1.3, for $O \neq A \in \mathcal{K}$, there is a unique $\delta_A \in \mathcal{D}_\alpha(O)$ such that $\delta_A(E) = A$, where δ_A is of the form $\sigma_g\sigma_h$ for $g,h \in O_\alpha$. From Axiom 9 and the proof of 3.5.1.1, for any $r \in O_\alpha$, $\sigma_g\sigma_h\sigma_r = \sigma_w$ for a unique $w \in O_\alpha$. That is, for every fixed $r \in O_\alpha$, every $O \neq A \in \mathcal{K}$ can be uniquely writen in the form $\delta_A = \sigma_w\sigma_r$ where $\sigma_w(E^r) = A$. (The uniqueness follows from the fact that $(A^r)^{w_1} = (A^r)^{w_2}$ implies $w_1 = w_2$ if $A^r \neq A$ [30].)

Therefore, for every $O \neq A \in \mathcal{K}$, there is a unique $a \in O_\alpha$, such that $E^{ta} = A$. That is, there is a unique $a \in O_\alpha$, such that $\sigma_a\sigma_t = \delta_A$. Also, every $P \in \mathfrak{X}_\alpha$, $P|\alpha$, can be uniquely writen as $P = P_1O\overline{P}_2$, where $P_1 \in \mathcal{K}$ and $\bar{P}_2 \in \mathcal{K}'$.

Let $P_2 = \overline{P_2}^t \in \mathcal{K}$. Then we may uniquely write $P = P_1OP_2^t$, where $P_1, P_2 \in \mathcal{K}$. Define the map $\hat{\delta}_A : \mathfrak{X}_\alpha \mapsto \mathfrak{X}_\alpha$, by $\hat{\delta}_A(P) = P_1^{ta}O(P_2^{ta})^t$, for $O \neq A \in \mathcal{K}$, $P|\alpha$.

Theorem 3.6.5. *The map $\hat{\delta}_A : \mathfrak{X}_\alpha \mapsto \mathfrak{X}_\alpha$ is a dilation on $\mathfrak{X}_\alpha$ with fixed point O and dilation factor A, for all $O \neq A \in \mathcal{K}$.*

Proof: If $P_1^{ta}OP_2^{tat} = Q_1^{ta}OQ_2^{tat}$, then by unicity, $P_1^{ta} = Q_1^{ta}$. So $P_1 = Q_1$, $P_2^{tat} = Q_2^{tat}$, and $P_2 = Q_2$. Hence, $P = P_1OP_2^t = Q_1OQ_2^t = Q$, and $\hat{\delta}_A$ is injective. Let $Q|\alpha$ with $Q = Q_1OQ_2^t$; $Q_1, Q_2 \in \mathcal{K}$. Let $P_1 = Q_1^{at} \in \mathcal{K}$ and $P_2 = Q_2^{at} \in \mathcal{K}$. Then $P = P_1OP_2^t|\alpha$ and $\hat{\delta}_A(P) = P_1^{ta}OP_2^{tat} = (Q_1^{at})^{ta}O(Q_2^{at})^{tat} = Q_1OQ_2^t = Q$. Therefore, $\hat{\delta}_A$ is onto.

We claim that if $P \neq Q$, then $g_{PQ} \parallel g_{\hat{\delta}_A(P)\hat{\delta}_A(Q)}$. First we show that $\hat{\delta}_A(P) \in g_{OP}$. If $P \in \mathcal{K}$, then $P = POO^t$ and $\delta_A(P) = P^{ta}OO^{tat} = P^{ta}OO = P^{ta} \in \mathcal{K}$. If $P \in \mathcal{K}'$, then putting $R = P^t \in \mathcal{K}$, we have $P = OOR^t$ and $\hat{\delta}_A(P) = O^{ta}OR^{tat} =$
$= OOR^{tat} = R^{tat} = P^{at} \in \mathcal{K}'$.

So suppose that $g_{OP} \equiv g$ is nonisotropic and write $P = P_1OP_2^t$, where $P_1 = P_2^{tg}$. If $O \in g$, then $XOY \in g \leftrightarrow X = Y^g$, where $X \in \mathcal{K}$ and $Y \in \mathcal{K}'$. Thus, $\hat{\delta}_A(P) = P_1^{ta}OP_2^{tat}$ and $P_2^{tat} = P_1^{gat} = P_1^{tag}$, which implies that $\hat{\delta}_A(P) \in g$.

Claim. $\hat{\delta}_A(POQ) = \hat{\delta}_A(P)O\delta_A(Q)$, for all $P, Q \in \mathfrak{X}_\alpha$. Let $P = P_1OP_2^t$ and $Q = Q_1OQ_2^t$. Then $POQ = P_1OP_2^tOQ_1OQ_2^t = (P_1OQ_1)O(P_2OQ_2)^t$, with $P_1OQ_1 \in \mathcal{K}$ and $(P_2OQ_2)^t \in \mathcal{K}'$. Therefore,

$$\begin{aligned}\hat{\delta}_A(POQ) &= (P_1OQ_1)^{ta}O(P_2OQ_2)^{tat} = P_1^{ta}OQ_1^{ta}OP_2^{tat}OQ_2^{tat} = (P_1^{ta}OP_2^{tat})O(Q_1^{ta}OQ_2^{tat}) \\ &= \hat{\delta}_A(P)O\hat{\delta}_A(Q).\end{aligned}$$

The claim follows.

Proof of (iii): Assume that $P, O,$ and Q are collinear. Then there is a line g, such that $P, O, Q \in g$. Thus, $\hat{\delta}_A(P), \hat{\delta}_A(Q) \in g$ and $g_{PQ} = g \parallel g = g_{\hat{\delta}_A(P)\hat{\delta}_A(Q)}$.

Conversely, suppose that $P, Q,$ and O are noncollinear. Let $g_1 = g_{OP}$, $g_2 = g_{OQ}$, and $g_3 = g_{O,POQ}$. Then $g_1, g_2,$ and g_3 are distinct lines through O. Indeed, if $g_3 = g_1$, say, then $POQ \in g_{OP}$ and we would have $OP(POQ) = Q \in g_{OP}$, which contradicts our assumption of noncollinearity. Now,

$$\begin{gathered}P(POQ) = OQ \;\text{ and }\; \hat{\delta}_A(P)(\hat{\delta}_A(POQ)) = \hat{\delta}_A(\hat{\delta}_A(P)O\hat{\delta}_A(Q)) = O\hat{\delta}_A(Q) \\ g_{P,POQ} \parallel g_{OQ} = g_{O\hat{\delta}_A(Q)} \parallel g_{\hat{\delta}_A(P)\hat{\delta}_A(POQ)}, \text{ and } g_{P,POQ} \parallel g_{\hat{\delta}_A(P)\hat{\delta}_A(POQ)}.\end{gathered}$$

Similarly, $Q(POQ) = Q(QOP) = OP$; and $\hat{\delta}_A(Q)(\hat{\delta}_A(POQ)) = \hat{\delta}_A(\hat{\delta}_A(Q)O\hat{\delta}_A(P)) = O\hat{\delta}_A(P)$. This implies that $g_{Q,POQ} \parallel g_{OP}$ and $g_{OP} = g_{O\hat{\delta}_A(P)} \parallel g_{\hat{\delta}_A(Q)\hat{\delta}_A(POQ)}$. Hence, $g_{Q,POQ} \parallel g_{\hat{\delta}_A(Q)\hat{\delta}_A(POQ)}$. By Axiom D, it follows that $g_{PQ} \parallel g_{\hat{\delta}_A(P)\hat{\delta}_A(Q)}$. Therefore, $\hat{\delta}_A : \mathfrak{X}_\alpha \mapsto \mathfrak{X}_\alpha$ is a dilation on $\mathfrak{X}_\alpha$. ■

Now we show that $\hat{\delta}_A = \bar{\delta}_A$ on $\mathfrak{X}_\alpha$. Because $\hat{\delta}_A$ and $\overline{\delta}_A$ are dilations on $\mathfrak{X}_\alpha$, a dilation is uniquely determined by the images of two points,and $\hat{\delta}_A(O) = O = \bar{\delta}_A(O)$, then it suffices to show that $\hat{\delta}_A(E) = \bar{\delta}_A(E)$. By definition of $\hat{\delta}_A$, $\hat{\delta}_A(E) = A$. By Lemma 3.6.3, $\bar{\delta}_A(E) = A$.

To extend the above idea to any $\eta \in \mathcal{M}$ such that $t \subset \mathfrak{X}_\eta$, let $\alpha \neq \eta \in \mathcal{M}$ such that $t \subset \mathfrak{X}_\eta$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the isotropic lines in $\mathfrak{X}_\eta$ through O. Because t is nonisotropic, then by Axiom 14 there exist γ, δ such that $\eta = \gamma\delta$ and $t = [\eta,\gamma,\delta]$. Thus $\sigma_t : \mathfrak{X}_\eta \mapsto \mathfrak{X}_\eta$ is well-defined. Every $B \in t \subset \mathfrak{X}_\eta$ may be uniquely written as $B = B_1OB_2$, where $B_i \in \mathcal{K}_i$. Because $O \in t$ and t is nonisotropic, then $B_2 = B_1^t$. Thus, $B = B_1OB_2^t$ where $B_1 \in \mathcal{K}_1$. So, in particular, there are unique $E_1, A_1 \in \mathcal{K}_1$ such that $E_t = E_1OE_1^t$ and $A_t = A_1OA_1^t$, for $A_t \in t$. As before, there is a unique $a_1 \in O_\eta$ such that $E_1^{ta_1} = A_1$. Hence, every $X \in \mathfrak{X}_\eta$ can be uniquely written as $X = X_1OX_2^t$, where $X_1, X_2 \in \mathcal{K}_1 \subset \mathfrak{X}_\eta$. This defines a map $\delta_{A\eta} : \mathfrak{X}_\eta \mapsto \mathfrak{X}_\eta$, given by

$$\delta_{A\eta}(X) = X_1^{ta_1}OX_2^{ta_1t}, \quad \text{for } X \in \mathfrak{X}_\eta.$$

Proposition 3.6.6. *Let* $\delta_{A\eta} : \mathfrak{X}_\eta \mapsto \mathfrak{X}_\eta$ *be the map defined above. Then*

1. $\delta_{A\eta}$ *is a dilation on* $\mathfrak{X}_\eta$.
2. $\delta_{A\eta} = \bar{\delta}_A$ *on* $\mathfrak{X}_\eta$.
3. *if* $P, O,$ *and* Q *are collinear points in* $\mathfrak{X}_\eta$, *not necessarily distinct, then* $\delta_{A\eta}(POQ) = \delta_{A\eta}(P)O\delta_{A\eta}(Q)$.
4. *Moreover,* $\bar{\delta}_A(POQ) = \bar{\delta}_A(P)O\bar{\delta}_A(Q)$, *for every* $P, Q | \eta$ *such that* $O, P,$ *and* Q *are collinear.* ■

To obtain a scalar multiplication on $\mathcal{V}$, for all $O \neq A \in \mathcal{K}$ and all $OX \in \mathcal{V}$, define

$$A \cdot OX \equiv \bar{\delta}_A(O)\bar{\delta}_A(X) = O\bar{\delta}_A(X) \text{ and } O \cdot OX = 1_{\mathfrak{G}}.$$

We now verify the vector space properties.

Lemma 3.6.7. *If* $A, A' \in \mathcal{K}$ *and* $OP \in \mathcal{V}$, *then* $(A + A') \cdot OP = A \cdot OP + A' \cdot OP$.

Proof: Suppose $P \in t$ and $P \neq O$, and recall $t, \mathcal{K} \subset \mathfrak{X}_\alpha$. Let $P, E | \alpha, \beta$; $A | \gamma$ with $\gamma \parallel \beta$; $A' | \gamma'$, $\gamma' \parallel \beta$; and $O | \delta$ with $\delta \parallel \beta$. Then $\gamma \parallel \gamma' \parallel \delta$; $g_1 = [\alpha,\gamma] \parallel g_{EP}$ with $g_1 \subset \mathfrak{X}_\alpha$; $g_2 = [\alpha,\gamma'] \parallel g_{EP}$ with $g_2 \subset \mathfrak{X}_\alpha$; and $g_3 = [\alpha,\delta] \parallel g_{EP}$ with $g_3 \subset \mathfrak{X}_\alpha$. Thus, $\bar{\delta}_A(P) \in g_1 \cap t$ and $\bar{\delta}_{A'}(P) \in g_2 \cap t$. Thus, $\bar{\delta}_A(P) | \alpha, \gamma$ and $\bar{\delta}_{A'}(P) | \alpha, \gamma'$. It follows that

$\bar{\delta}_A(P)O\bar{\delta}_{A'}(P)|\alpha,\gamma\delta\gamma'$; $\bar{\delta}_A(P)O\bar{\delta}_{A'}(P) \in t$; $\gamma\delta\gamma' = \varepsilon \parallel \beta$; and $AOA'|\alpha,\gamma\delta\gamma' = \varepsilon$. Therefore, $AOA' \in [\alpha,\varepsilon] \parallel g_{EP}$ and $\bar{\delta}_A(P)O\bar{\delta}_{A'}(P) \in t \cap [\alpha,\varepsilon]$. Hence, $\bar{\delta}_{AOA'}(P) = \bar{\delta}_A(P)O\bar{\delta}_{A'}(P)$.

Suppose $P \notin t$. Because $g_{OP}, t \subset \mathfrak{X}_\eta$, then replacing E with E_t, A with A_t, A' with A'_t, and α with η in the first part of the proof and the result follows. ■

Lemma 3.6.8. *If $A \in \mathcal{K}$ and $OP, OQ \in \mathcal{V}$ then $A \cdot (OP + OQ) = A \cdot OP + A \cdot OQ$.*

Proof: We need to show that $\bar{\delta}_A(POQ) = \bar{\delta}_A(P)O\bar{\delta}_A(Q)$. Suppose P, O, and Q are collinear. Let $g = g_{OP} = g_{OQ} = g_{O,POQ}$ and $\eta \in \mathcal{M}$ such that $g, t \subset \mathfrak{X}_\eta$. Then the result follows from Proposition 3.6.6(iv).

Conversely, suppose that P, O, and Q are not collinear. Because P, O, and Q are not collinear then $P \neq Q$, and from Lemma 3.6.4 it follows that $g_{PQ} \parallel g_{\bar{\delta}_A(P)\bar{\delta}_A(Q)}$. Applying Axiom R we obtain $g_{O,POQ} = g_{O\bar{\delta}_A(P)\bar{\delta}_A(Q)}$. That is, $\bar{\delta}_A(P)O\bar{\delta}_A(Q) \in g_{O,POQ}$ by construction. Again by Lemma 3.6.4,

$$g_{\bar{\delta}_A(P)\bar{\delta}_A(POQ)} \parallel g_{O,POQ} \quad \text{and} \quad \bar{\delta}_A(P)\bar{\delta}_A(P)O\bar{\delta}_A(Q) = O\bar{\delta}_A(Q).$$

This implies that $g_{\bar{\delta}_A(P),\bar{\delta}_A(P)O\bar{\delta}_A(Q)} \parallel g_{O\bar{\delta}_A(Q)} = g_{OQ} \parallel g_{P,POQ}$, as $P(POQ) = OQ$. Thus, $g_{\bar{\delta}_A(P),\bar{\delta}_A(P)O\bar{\delta}_A(Q)} = g_{\bar{\delta}_A(P)\bar{\delta}_A(POQ)}$ because both lines contain $\bar{\delta}_A(P)$ and are parallel to $g_{P,POQ}$. Since

$g_{\bar{\delta}_A(P),\bar{\delta}_A(P)O\bar{\delta}_A(Q)} \cap g_{P,POQ} = \{\bar{\delta}_A(P)O\bar{\delta}_A(Q)\}$ and $g_{\bar{\delta}_A(P)\bar{\delta}_A(POQ)} \cap g_{O,POQ} = \{\bar{\delta}_A(POQ)\}$, then $\bar{\delta}_A(P)O\bar{\delta}_A(Q) = \bar{\delta}_A(POQ)$. ■

Lemma 3.6.9. *If $A, A' \in \mathcal{K}$ and $OP \in \mathcal{V}$ then $A \cdot (A' \cdot OP) = (A \cdot A') \cdot OP$.*

Proof: We need to show that $\bar{\delta}_A(\bar{\delta}_{A'}(P)) = \bar{\delta}_{A \cdot A'}(P)$; that is, $\bar{\delta}_A \circ \bar{\delta}_{A'} = \bar{\delta}_{A \cdot A'}$. Now $\bar{\delta}_A \circ \bar{\delta}_{A'}$ and $\bar{\delta}_{A \cdot A'}$ are dilations on $\mathfrak{X}$ and $\bar{\delta}_{A \cdot A'}(O) = O$ by construction, so it suffices to show that $(\bar{\delta}_A \circ \bar{\delta}_{A'})(E) = \bar{\delta}_{A \cdot A'}(E)$ and $(\bar{\delta}_A \circ \bar{\delta}_{A'})(O) = \bar{\delta}_{A \cdot A'}(O)$. Now $E \in \mathcal{K} \subset \mathfrak{X}_\alpha$, so on $\mathfrak{X}_\alpha$, $\bar{\delta}_A \circ \bar{\delta}_{A'} = \hat{\delta}_A \circ \hat{\delta}_{A'}$ and $\bar{\delta}_{A \cdot A'} = \hat{\delta}_{A \cdot A'}$. But on $\mathcal{K}$, by Lemma 3.5.11, $\hat{\delta}_A \circ \hat{\delta}_{A'} = \hat{\delta}_{A \cdot A'}$. Therefore, $(\bar{\delta}_A \circ \bar{\delta}_{A'})(E) = A \cdot (A' \cdot E) = (A \cdot A') \cdot E = \bar{\delta}_{A \cdot A'}(E)$. ■

Lemma 3.6.10. *For $E \in \mathcal{K}$, the multiplicative unit, and $OP \in \mathcal{V}$, $E \cdot OP = OP$.*

Proof: We need to show that $\bar{\delta}_E = 1_{\mathfrak{X}}$. Now $1_{\mathfrak{X}}$ is clearly a dilation on $\mathfrak{X}$ and $1_{\mathfrak{X}}(O) = O = \bar{\delta}_E(O)$. Thus, $1_{\mathfrak{X}}(E) = E = E \cdot E = \hat{\delta}_E(E) = \bar{\delta}_E(E)$. ■

Theorem 3.6.11. *The space $(\mathcal{V}, \mathcal{K})$ constructed above is a vector space.* ■

The triple $(\mathfrak{X}, \mathcal{V}, \mathcal{K})$, is an affine space.[23,p.6] A set $\mathfrak{X}$ along with a vector space $\mathcal{V}$ over a field $\mathcal{K}$ is an *affine space* if for every $\bar{v} \in \mathcal{V}$ and for every $X \in \mathfrak{X}$, there is defined a point $\bar{v}X \in \mathfrak{X}$ such that the following conditions hold.

1. If $\bar{v}, \bar{w} \in \mathcal{V}$ and $X \in \mathfrak{X}$, then $(\bar{v} + \bar{w})X = \bar{v}(\bar{w}X)$.
2. If $\bar{0}$ denotes the zero vector, $\bar{0}X = X$ for all $X \in \mathfrak{X}$.
3. For every ordered pair (X, Y) of points of $\mathfrak{X}$, there is one and only one vector $\bar{v} \in \mathcal{V}$ such that $\bar{v}X = Y$.

The dimension n of the vector space $\mathcal{V}$ is also called the dimension of the affine space $\mathfrak{X}$.

Theorem 3.6.12. *$(\mathfrak{X}, \mathcal{V}, \mathcal{K})$ is an affine space.*

Proof: If $OV, OW \in \mathcal{V}$ and $X \in \mathfrak{X}$, we have

(i) $(OV)X = OVX \in \mathfrak{X}$.

(ii) $(OV + OW)X = (OVOW)X = OV(OWX)$.

(iii) $OOX = 1_{\mathfrak{G}}X = X$.

(iv) for $Y \in \mathfrak{X}$, $OYX = Z \in \mathfrak{X}$ and $(OZ)X = Y$.

Now if $OPX = Y$, then $OPX = OZX$, $P = Z$, and $OP = OZ$. ■

3.7 Subspaces and Dimensions

In this section we show that our lines and planes have the proper dimensions. We are then able to conclude that $(\mathcal{V}, \mathcal{K})$ and $(\mathfrak{X}, \mathcal{V}, \mathcal{K})$ are four-dimensional spaces.

Proposition 3.7.1 *Let g be any line through O and put $\hat{g}(O) = \{OA : A \in g\}$. Then $\hat{g}(O)$ is a one dimensional subspace of $\mathcal{V}$.*

Proof: First note that $1_{\mathfrak{G}} = OO \in \hat{g}(O)$, so the zero vector is in $\hat{g}(O)$. Let $A, B \in g$. Then $C = AOB \in g$ by 3.1.4.7 and $OA + OB = OAOB = OC \in \hat{g}(O)$. From Section 3.6,

$\bar{\delta}_A(B) \in g$, for all A in $\mathcal{K}$ and all B in g. So $A \cdot OB \in \hat{g}(O)$ for all $A \in \mathcal{K}$ and $OB \in \hat{g}(O)$. Hence, $\hat{g}(O)$ is a subspace of $\mathcal{V}$.

It must now be shown that the dimension of $\hat{g}(O)$ is one. If $g = t$, then $\hat{g}(O) = \langle OE_t \rangle$ because for every $A_t \in t$, $A_t = \bar{\delta}_A(E_t)$ by Lemma 3.6.3. So suppose that $g \neq t$ and fix $B \in g$. Let $h = g_{BE_t}$. Then for all $O \neq D \in g$, there ia a unique $d$such that $D \in d$, $d \parallel h$, and $d \cap t \neq \emptyset$. Put $\{F_t\} = d \cap t$. Then for $F_t = FOF^t$ with $F \in \mathcal{K}$ it follows that

$$\bar{\delta}_F(B) = D \qquad \text{and} \qquad OD = F \cdot OB.$$

Hence, $\hat{g}(O) = \langle OB \rangle$. ■

Corollary 3.7.2. *Following the terminology of Snapper and Troyer* [23, p.11],

$$g = S(O, \hat{g}(O)) \equiv \{A = O(OA) : OA \in \hat{g}(O)\}$$

is an affine subspace of dimension one.

Proposition 3.7.3. *Let* $\alpha \in \mathcal{P}$ *with* $O|\alpha$ *and put* $\hat{\mathfrak{X}}_\alpha(O) = \{OA : A|\alpha\}$. *Then* $\hat{\mathfrak{X}}_\alpha(O)$ *is a two dimensional subspace of* $\mathcal{V}$.

Proof: Clearly, $1_{\mathfrak{G}} = OO \in \hat{\mathfrak{X}}_\alpha(O)$, so the zero vector is in $\hat{\mathfrak{X}}_\alpha(O)$. Let $C, D|\alpha$ and $A, B \in \mathcal{K}$. Then by 3.1.4.7 we have $COD = F|\alpha$ and by Lemma 3.6.3,

$$\bar{\delta}_A(C) \in g_{OC} \subset \mathfrak{X}_\alpha \qquad \text{and} \qquad \bar{\delta}_B(D) \in g_{OD} \subset \mathfrak{X}_\alpha$$

so that $\bar{\delta}_A(C)O\bar{\delta}_B(D)|\alpha$. It follows that $OC + OD = OCOD = OF \in \hat{\mathfrak{X}}_\alpha(O)$, and

$$A \cdot OC + B \cdot OD = O\bar{\delta}_A(C) + O\bar{\delta}_B(D) = O(\bar{\delta}_A(C)O\bar{\delta}_B(D) \in \hat{\mathfrak{X}}_\alpha(O).$$

Hence, $\hat{\mathfrak{X}}_\alpha(O)$ is a subspace of $\mathcal{V}$.

Thus, it remains to show that $\hat{\mathfrak{X}}_\alpha(O)$ is two dimensional. There are two cases: $\alpha \in \mathcal{M}$ and $\alpha \in \mathcal{G}$. Suppose first that $\alpha \in \mathcal{M}$. We construct a basis for $\hat{\mathfrak{X}}_\alpha(O)$ using isotropic lines. To this end, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the isotropic lines in $\mathfrak{X}_\alpha$ through O. For any $P|\alpha$ we may uniquely write in $\mathfrak{X}_\alpha$, $P = P_1OP_2$, with $P_1 \in \mathcal{K}_1$ and $P_2 \in \mathcal{K}_2$.

From Proposition 3.7.1 above, $\hat{\mathcal{K}}_1(O) = \langle OB \rangle$ for any $O \neq B \in \mathcal{K}_1$ and $\hat{\mathcal{K}}_2(O) = \langle OC \rangle$ for any $O \neq C \in \mathcal{K}_2$. From this it follows that $P_1 = \bar{\delta}_A(B)$ and $P_2 = \bar{\delta}_{A'}(C)$ for some $A, A' \in \mathcal{K}$. Thus,

$$OP = A \cdot OB + A' \cdot OC$$

and $\{OC, OB\}$ span $\hat{\mathfrak{X}}_\alpha(O)$. Now if $A \cdot OB + A' \cdot OC = 1_{\mathfrak{G}}$, then $O\bar{\delta}_A(B)O\bar{\delta}_{A'}(C) = 1_{\mathfrak{G}}$. This implies that $\bar{\delta}_A(B)O\bar{\delta}_{A'}(C) = O$ and $O\bar{\delta}_{A'}(C) = \bar{\delta}_A(B)O$. Because $O, \bar{\delta}_{A'}(C) \in \mathcal{K}_2$ and $O, \bar{\delta}_A(B) \in \mathcal{K}_1$, then either $\mathcal{K}_1 \parallel \mathcal{K}_2$ or $O\bar{\delta}_{A'}(C) = 1_{\mathfrak{G}} = \bar{\delta}_A(B)O$. But $\mathcal{K}_1 \nparallel \mathcal{K}_2$, so $\bar{\delta}_{A'}(C) = \bar{\delta}_A(B) = O$. Because $\alpha \in \mathcal{M}$, then from 3.3 and 3.6 there exists $t', a' \in O_\alpha$ such that $\bar{\delta}_{A'}(C) = C^{t'a'} = O$. This implies that $C = O^{a't'} = O$ or $A' = O$. By assumption $C \neq O$ and $D \neq O$, thus it must be the case that $A = A' = O$. Hence, $\{OC, OB\}$ is a linearly independent set in $\hat{\mathfrak{X}}_\alpha(O)$ and $\hat{\mathfrak{X}}_\alpha(O)$ is two dimensional.

Suppose now that $\alpha \in \mathcal{G}$. We construct an "orthogonal" basis. By 3.1.7.2, there exist $\beta, \gamma \in \mathcal{G}$ such that $O = \alpha\beta\gamma$ and $\alpha \perp \beta \perp \gamma \perp \alpha$. Let $x = [\alpha, \beta, \alpha\beta]$ and $y = [\alpha, \gamma, \alpha\gamma]$. Note that if $x = y$ then by Axiom 14, $\alpha\beta = \gamma$ and $O = \alpha\beta\gamma = 1_{\mathfrak{G}}$, so $x \neq y$. Let $P|\alpha$ and let $\beta', \gamma'|P$ with $\beta' \perp \beta$ and $\gamma' \perp \gamma$. Put $P_1 = \beta\beta'$ and $P_2 = \gamma\gamma'$. Now $P|\alpha, \beta'$; $\alpha \perp \beta$; $\beta' \perp \beta$, so that $\alpha \perp \beta'$ by 3.1.6.18. Because $P|\alpha, \gamma'$; $\alpha \perp \gamma$; $\gamma' \perp \gamma$, then $\alpha \perp \gamma'$. Thus,

$$P_1^\alpha = (\beta\beta')^\alpha = \beta^\alpha\beta'^\alpha = \beta\beta' = P_1 \text{ and } P_2^\alpha = (\gamma\gamma')^\alpha = \gamma\gamma' = P_2,$$

so $P_1, P_2|\alpha$. This implies that $P_1|\alpha, \beta$ and $P_2|\alpha, \gamma$ so $P_1 \in x$ and $P_2 \in y$. If $Q = P_1OP_2$ then $Q = P_1OP_2 = \beta'\beta\beta\alpha\gamma\gamma\gamma' = \beta'\alpha\gamma'$. Because $\beta' \perp \alpha$, let $\delta = \alpha\beta' \in \mathcal{P}$. Then $Q = \delta\gamma'$ and $\delta \perp \gamma'$. Thus $P|\beta', \alpha$ implies that $P|\beta'\alpha = \delta$. Thus we have $P, Q|\delta, \gamma'$ with $\delta \perp \gamma'$, and $P = Q$ by Axiom 3. That is, $P = P_1OP_2$ with $P_1 \in x$ and $P_2 \in y$.

Let $O \neq X \in x$ and $O \neq Y \in y$. Then from Proposition 3.7.1 above we have $x = \langle OX \rangle$ and $y = \langle OY \rangle$ and there exist $A, B \in \mathcal{K}$ such that $OP = A \cdot OX + B \cdot OY$, where

$P_1 = \bar{\delta}_A(X)$ and $P_2 = \bar{\delta}_B(Y)$. Hence, $\{OX, OY\}$ spans $\hat{\mathfrak{X}}_\alpha(O)$. If $A \cdot OX + B \cdot OY = 1_{\mathfrak{G}}$, then we obtain $O\bar{\delta}_A(X) = \bar{\delta}_B(Y)O$. Because $x \not\parallel y$, it follows that $\bar{\delta}_A(X) = O = \bar{\delta}_B(Y)$. Let $\mathfrak{X}_\eta$ be the unique plane containing t and x and $\mathcal{K}_1$ and $\mathcal{K}_2$ the isotropic lines in $\mathfrak{X}_\eta$ through O . Then we may write $\bar{\delta}_A(X) = X_1^{ta}OX_1^{\prime tat}$, where $X_1 \in \mathcal{K}_1$, and $X_1' \in \mathcal{K}_2$. As above it follows that $A = O$ and similarly, $B = O$. Therefore, $\hat{\mathfrak{X}}_\alpha(O)$ is two dimensional. ■

Corollary 3.7.4. *$\mathfrak{X}_\alpha = S(O, \hat{\mathfrak{X}}_\alpha(O))$ is an affine subspace of dimension two for all $\alpha \in \mathcal{P}$, $O|\alpha$.*

Theorem 3.7.5. *$(\mathcal{V},\mathcal{K})$ is a four-dimensional vector space and hence, $(\mathfrak{X}, \mathcal{V}, \mathcal{K})$ is a four-dimensional affine space.*

Proof: Let $OP \in \mathcal{V}$ and let $O = \alpha\beta$ with $\alpha \in \mathcal{M}$ and $\beta \in \mathcal{G}$. Let $P|\alpha', \beta'$ with $\alpha' \perp \alpha$ and $\beta' \perp \beta$. Put $P' = \alpha'\alpha$ and $P'' = \beta'\beta$. Then $Q = P'OP'' = \alpha'\alpha\alpha\beta\beta\beta' = \alpha'\beta'$. Thus, $P, Q|\alpha', \beta'$ with $\alpha' \perp \beta'$. Therefore $P = Q = P'OP''$ with $P'|\alpha$ and $P''|\beta$. Since $\hat{\mathfrak{X}}_\alpha(O)$ and $\hat{\mathfrak{X}}_\beta(O)$ are two-dimensional then there exist bases $\{OZ, OT\} \subset \hat{\mathfrak{X}}_\alpha(O)$ and $\{OX, OY\} \subset \hat{\mathfrak{X}}_\beta(O)$ such that

$$OP' = A \cdot OZ + A' \cdot OT \quad \text{and} \quad OP'' = B \cdot OX + B' \cdot OY$$

for some $A, A', B, B' \in \mathcal{K}$. Thus, $P = A \cdot OZ + A' \cdot OT + B \cdot OX + B' \cdot OY$ and $\{OX, OY, OZ, OT\}$ span $\mathcal{V}$. If $A \cdot OZ + A' \cdot OT + B \cdot OX + B' \cdot OY = 1_{\mathfrak{G}}$, then in particular, $OP = OP' + OP'' = 1_{\mathfrak{G}}$. This implies that $OP'' = P'O$. So either $g_{OP''} \parallel g_{P'O}$ or $P'' = P' = O$. But $g_{OP''} \not\parallel g_{OP'}$ and therefore, $A \cdot OZ + A' \cdot OT = OP' = 1_{\mathfrak{G}} = OP'' = B \cdot OX + B' \cdot OY$. As was shown in Proposition 3.7.2, we obtain $A = A' = B = B' = O$ and the result follows. ■

3.8 Orthogonality

In this section we extend the definition of orthogonality to include lines and then use this definition to define orthogonal vectors.

Definition 3.8.1. Let g and h be two lines. We say that g is *perpendicular to or orthogonal to* h, denoted by $g \perp h$, if there exist $\alpha, \beta \in \mathcal{P}$ such that $g \subset \mathfrak{X}_\alpha$, $h \subset \mathfrak{X}_\beta$, $\alpha \mathbin{\rotatebox[origin=c]{180}{$\top$}} \beta$, and $P = \alpha\beta \in g,h$. In this case, P is the point of intersection of g and h.

Lemma 3.8.2. *If* g *and* h *are isotropic then* g *is not orthogonal to* h.

Proof: This follows directly from our definition above and from our definition of $\alpha \mathbin{\rotatebox[origin=c]{180}{$\top$}} \beta$. For if $\alpha \mathbin{\rotatebox[origin=c]{180}{$\top$}} \beta$ then one of α and β must be in $\mathcal{G}$ and by Axiom 12, all lines in a plane $\mathfrak{X}_\beta$ for $\beta \in \mathcal{G}$ are nonisotropic. ■

In Minkowski space, if g and h are two isotropic lines then $g \perp h \leftrightarrow g \parallel h$. Thus we extend the above definition in the following way.

Definition 3.8.3. If g and h are isotropic lines, then $g \perp h \leftrightarrow g \parallel h$.

Definition 3.8.4. If g is a line and $\alpha \in \mathcal{P}$ then we say that g *is orthogonal to or perpendicular to* $\mathfrak{X}_\alpha$, $g \perp \mathfrak{X}_\alpha$, if there exists a $\beta \in \mathcal{P}$ such that $g \subset \mathfrak{X}_\beta$, $\beta \mathbin{\rotatebox[origin=c]{180}{$\top$}} \alpha$, and $P = \alpha\beta \in g$. In this case, $P = \alpha\beta$ is the point of intersection of g and $\mathfrak{X}_\alpha$.

Definition 3.8.5. For every $1_{\mathfrak{G}} \neq OA, OB \in \mathcal{V}$, we say that *$OA$ is orthogonal to OB*, $OA \perp OB$, if and only if $g_{OA} \perp g_{OB}$; that is, there exist $\alpha, \beta \in \mathcal{P}$ such that $g_{OA} \subset \mathfrak{X}_\alpha$, $g_{OB} \subset \mathfrak{X}_\beta$, and $O = \alpha\beta$. For the zero vector $1_{\mathfrak{G}} = OO$, we define $1_{\mathfrak{G}} \perp OA$, for all $OA \in \mathcal{V}$.

Lemma 3.8.6. *From* 3.1.2.1 *and* 3.1.2.2 *it follows that for* $\xi \in \mathfrak{G}$:

(i) $g \perp h \leftrightarrow g^\xi \perp h^\xi$.

(ii) $g \perp \mathfrak{X}_\alpha \leftrightarrow g^\xi \perp \mathfrak{X}_{\alpha^\xi}$.

(iii) $OA \perp OB \leftrightarrow (OA)^\xi \perp (OB)^\xi$. ■

Lemma 3.8.7. *If* g *is a nonisotropic line then* g *is not orthogonal to* g.

Proof: If $g \perp g$ then there exist $\alpha, \beta \in \mathcal{P}$ such that $g \subset \mathfrak{X}_\alpha, \mathfrak{X}_\beta$; $\alpha \mathbin{\rotatebox[origin=c]{180}{$\top$}} \beta$, and $P = \alpha\beta \in g$. But for every point $Q \in g$, $Q|\alpha,\beta$ with $\alpha \mathbin{\rotatebox[origin=c]{180}{$\top$}} \beta$ which implies that $P = Q$ by Axiom 3; that is, g is a line which contains only one point, which contradicts the definition of a line. ■

Additional axioms and their immediate consequences. To complete our preparations for defining our polarity and thus obtaining the Minkowski metric, we recall our final three axioms.

Axiom U. *($U^{\perp}$ subspace axiom) Let $O, A, B,$ and C be any four, not necessarily distinct, points with $A, O | \alpha$; $O, B | \beta$; $O, C | \gamma, \delta$ and $\alpha \bot \gamma$ and $\beta \bot \delta$. Then there exists $\lambda, \varepsilon \in \mathcal{P}$ such that $\lambda \bot \varepsilon$; $O, AOB | \lambda$; and $O, C | \varepsilon$.*

Axiom S1. *If $g \subset \mathfrak{X}_\alpha$, $\alpha \in \mathcal{G}$, $h \subset \mathfrak{X}_\beta$, $\beta \in \mathcal{G}$, and there exists $\gamma, \delta \in \mathcal{P}$ such that $\gamma \bot \delta$; $\gamma\delta \in g \cap h$; $g \subset \mathfrak{X}_\gamma$; and $h \subset \mathfrak{X}_\delta$ then there exists $\varepsilon \in \mathcal{G}$ such that $g, h \subset \mathfrak{X}_\varepsilon$. (If g and h are two orthogonal spacelike lines then there is a spacelike plane containing them.)*

Axiom S2. *Let g and h be two distinct lines such that $P \in g \cap h$ but there does not exist $\beta \in \mathcal{P}$ such that $g, h \subset \mathfrak{X}_\beta$. Then either there exists $\alpha, \gamma \in \mathcal{P}$ such that $\alpha \bot \gamma$, $g \subset \mathfrak{X}_\alpha$, and $h \subset \mathfrak{X}_\gamma$ or for all $A \in g$, there exists $B \in h$ such that P and APB are unjoinable.*

Lemma 3.8.8. *If $g, h \subset \mathfrak{X}_\alpha$ are nonisotropic for $\alpha \in \mathcal{P}$, then $g \perp h$ in the sense of Section* 3.3 *if and only if $g \perp h$ in the sense of Definition* 3.8.1.

Proof: Let $g = [\alpha, \beta, \delta]$, $h = [\alpha, \gamma, \lambda] \subset \mathfrak{X}_\alpha$ with $\alpha = \beta\delta = \gamma\lambda$. Recall that $g \perp h$ in the sense of 3.3 if , without loss of generality, $\beta \bot \gamma$ and $A = \beta\gamma \in g \cap h$. So, in particular, $A = \beta\gamma$ with $A \in g \cap h$ and $g \subset \mathfrak{X}_\beta$ and $h \subset \mathfrak{X}_\gamma$. So $g \perp h$ by 3.8.1.

Now suppose that there exist $\eta, \varepsilon \in \mathcal{P}$ such that $B = \eta\varepsilon \in g \cap h$ and $g \subset \mathfrak{X}_\eta$ and $h \subset \mathfrak{X}_\varepsilon$, but it is not the case that $\beta \bot \gamma$, nor that $\beta \bot \lambda$, nor that $\delta \bot \gamma$, nor that $\delta \bot \lambda$. Then by 3.3.8 there exists a unique $l \in \mathcal{L}_\alpha$ such that $B \in l$ and $l = [\alpha, \upsilon, \mu]$ for some $\upsilon, \mu \in \mathcal{P}$ with $\alpha = \upsilon\mu$, $\upsilon \bot \beta$, and $\mu \bot \delta$. If $l \neq h$ then $\hat{l}_\alpha(B)$ and $\hat{h}_\alpha(B)$ span $\hat{\mathfrak{X}}_\alpha(B)$. Thus if $C \in g$ there exists $L \in l$ and $H \in h$ such that $BLBH = BC$. By Axiom U, $BL \perp BC$ and $BH \perp BC$ imply that $BC = BLBH \perp BC$. That is, $g \perp g$, which cannot happen for nonisotropic g. Hence, $l = h$ and the result follows. ■

Lemma 3.8.9. *If* $\alpha \in \mathcal{P}$ *then for each* $P|\alpha$ *and for each nonisotropic line* $g \subset \mathfrak{X}_\alpha$ *there is a unique nonisotropic line* $h \subset \mathfrak{X}_\alpha$ *such that* $P \in h$ *and* $h \perp g$.

Proof: This follows directly from 3.3.8 and Lemma 3.8.8. ■

Lemma 3.8.10. *If* $g, h \subset \mathfrak{X}_\alpha$ *for* $\alpha \in \mathcal{P}$ *and* g, h *are nonisotropic, then* $g \perp h$ *if and only if* $\sigma_g\sigma_h = \sigma_h\sigma_g \neq 1_{\mathfrak{X}_\alpha}$.

Proof: This follows from 3.3.16 and Lemma 3.8.8. ■

Lemma 3.8.11. *Suppose that* $O \in c,d$; $c,d \subset \mathfrak{X}_\alpha$, $\alpha \in \mathcal{P}$ *with* c *and* d *nonisotropic and* $c \perp d$. *Let* $C \in c$ *and* $D \in d$, *then* $g_{O,DOC}$ *is not orthogonal to* c *if* $C, D \neq O$.

Proof: First note that $g_{O,DOC} \neq c$ or d because then we would have $DOC = D_1 \in d$, say, so that $C = ODD_1 \in d$ and $O \in d$ implies that $C = O$ or $c = d$. Now if $g_{O,DOC} \perp c$, then in $\mathfrak{X}_\alpha$ we have $\sigma_O = \sigma_c\sigma_d = \sigma_{g_{O,DOC}}\sigma_c$, which implies that $d = g_{O,DOC}$. ■

Lemma 3.8.12. *Let* $g,x \subset \mathfrak{X}_\alpha$ *with* g *isotropic,* $\alpha \in \mathcal{P}$, *and* $g \cap x = \{O\}$. *Then* g *is not orthogonal to* x *if* $g \neq x$.

Proof: If x is isotropic and $x \perp g$ then because $x \neq g$, there exists $\gamma, \delta \in \mathcal{P}$ such that $g \subset \mathfrak{X}_\gamma$, $x \subset \mathfrak{X}_\delta$, and $\gamma \perp \delta$. But then one of γ and δ must lie in $\mathcal{G}$. But no element in $\mathcal{G}$ can contain an isotropic line so x is not orthogonal to g.

Suppose that x is nonisotropic and let $O \in g \cap x$. Because g is isotropic then $\alpha \in \mathcal{M}$ and by 3.3.8, there is a unique nonisotropic line $h \subset \mathfrak{X}_\alpha$ such that $O \in h$ and $h \perp x$. Suppose that $g \perp x$ and let $O \neq K \in g$, $O \neq H \in h$, and $O \neq X \in x$. Then $OH \perp OX$ and $OK \perp OX$ so that by Axiom U, $O(HOK) = OHOK \perp OX$. If $g_{O,HOK}$ is nonisotropic then $g_{O,HOK} = h$. And because $g_{O,HOK} \subset \mathfrak{X}_\alpha$, then $HOK = H_1 \in h$. Then we have $K = OHH_1 \in h$ and $g = h$, a contradiction.

Suppose that $g_{O,HOK}$ is isotropic. If $g_{O,HOK} = g$ then we may write $HOK = K_1$ for some $K_1 \in g$. It follows that $H = K_1KO \in g$ and $g = h$. If $g_{O,HOK} \neq g$ then $g_{O,HOK}$ is the other isotropic line through O in $\mathfrak{X}_\alpha$. Because g and $g_{O,HOK}$ span $\mathfrak{X}_\alpha$ and

$g, g_{O,HOK} \perp x$ then by Axiom U, x is orthogonal to every line in $\mathfrak{X}_\alpha$ through O. So in particular, $x \perp h^x$ which implies that $x = h$. ■

Corollary 3.8.13. *If g is isotropic and $x \perp g$ then either $x = g$ or x is nonisotropic and x and g are noncoplanar; that is, there does not exist $\delta \in \mathcal{P}$ such that $x, g \subset \mathfrak{X}_\delta$.*

Lemma 3.8.14. *If g is isotropic, x is nonisotropic, $x \perp g$ with $x \neq g$, and $\{P\} = x \cap g$, then for all $P \neq A \in g$ and for all $P \neq B \in x$, g and g_{APB} are not coplanar.*

Proof: From Lemma 3.8.12, x and g are noncoplanar. Suppose that $A, O, AOB \mid \gamma$ for some $\gamma \in \mathcal{P}$. Then $B = OAAOB|\gamma$, which implies that $x = g_{OB} \subset \mathfrak{X}_\gamma$ and $g = g_{OA} \subset \mathfrak{X}_\gamma$, a contradiction to Lemma 3.8.11. ■

Lemma 3.8.15. *If $OC \in \mathcal{V}$ is isotropic and $OB \in \mathcal{V}$ is nonisotropic with $OC \perp OB$ then $OCOB \perp OC$.*

Proof: By Lemma 3.8.13, g_{OC} and g_{OB} are not coplanar and by Lemma 3.8.14, g_{OC} and $g_{O,COB}$ are not coplanar.By Axiom S2, either $g_{OC} \perp g_{OCOB}$ or there exists $D \in g_{OCOB}$ such that O and COD are unjoinable. So if g_{OC} is not orthogonal to g_{OCOB} then $g_{O,COD}$ is isotropic and by Axiom T, there exists a unique $\delta \in \mathcal{P}$ such that $g_{OC}, g_{O,COD} \subset \mathfrak{X}_\delta$. This implies that $O, C, COD|\delta$, $D = OCCOD|\delta$, $g_{OD} = g_{O,COD} \subset \mathfrak{X}_\delta$, and therefore, $g_{OC} \subset \mathfrak{X}_\delta$, a contradiction. ■

Lemma 3.8.16. *The zero vector, $1_{\mathfrak{G}} = OO$, is the only vector orthogonal to every vector in $\mathcal{V}$.*

Proof: This follows immediately from Lemma 3.8.11 above. ■

Theorem 3.8.17. *If $\mathcal{U}$ is a subspace of $\mathcal{V}$, then $\mathcal{U}^\perp \equiv \{OA \in \mathcal{V} : OA \perp OB, \ \forall OB \in \mathcal{U}\}$ is a subspace of $\mathcal{V}$.*

Proof: because the zero vector $1_{\mathfrak{G}}$ is orthogonal to every vector in $\mathcal{V}$ by definition, then $1_{\mathfrak{G}} \in \mathcal{U}^\perp$. Let $OA \in \mathcal{U}^\perp$ and $R \in \mathcal{K}$. because $R \bullet OA \in < OA >$, the subspace generated by OA, and $g_{OA} \perp g_{OB}$ for every $OB \in \mathcal{U}$ by the definitions of $\mathcal{U}^\perp$ and orthogonal vectors, then $R \bullet OA \in \mathcal{U}^\perp$.

Let $OA, OB \in \mathcal{U}^{\perp}$ and $OC \in \mathcal{U}$. If OC is nonisotropic then $OC \neq OA, OB$ and there exist $\alpha, \beta, \gamma, \delta \in \mathcal{P}$ such that

$$A, O|\alpha; \quad O, B|\beta; \quad O, C|\gamma, \delta; \quad \alpha \perp \gamma; \quad \text{and} \quad \beta \perp \delta.$$

By Axiom U, there exists $\lambda, \varepsilon \in \mathcal{P}$ such that $O = \lambda\varepsilon$, $AOB|\lambda$, and $O, C\ |\varepsilon$. Thus, $OA + OB = O(AOB) \perp OC$.

Now suppose that OC is isotropic. The following possibilities exist.

(i) If $OC \neq OA, OB$, then as in (a) above, $(OA + OB) \perp OC$.

(ii) If $OC = OA = OB$, then because OC is isotropic, $AOB = COC \in g_{OC}$ and $(OA + OB) \perp OC$.

(iii) If $OC = OA \neq OB$, then $OB \perp OC$ implies that OB is nonisotropic and $(OC + OB) = OCOB \perp OC$ by Lemma 3.8.14.

Hence, if $OA, OB \in \mathcal{U}^{\perp}$ then $OA + OB \in \mathcal{U}^{\perp}$ and $\mathcal{U}^{\perp}$ is a subspace of $\mathcal{V}$. ■

Theorem 3.8.18. *If $O = \alpha\beta$ then $\mathfrak{X}_{\alpha}(O)^{\perp} = \mathfrak{X}_{\beta}(O)$ and $\mathfrak{X}_{\beta}(O)^{\perp} = \mathfrak{X}_{\alpha}(O)$.*

Proof: From the definition of orthogonal vectors we clearly have $\mathfrak{X}_{\alpha}(O) \subset \mathfrak{X}_{\beta}(O)^{\perp}$ and $\mathfrak{X}_{\beta}(O)^{\perp} \subset \mathfrak{X}_{\alpha}(O)$. Now suppose that $OA \perp \mathfrak{X}_{\alpha}(O)$; that is, $g_{OA} \perp \alpha$. Then, there exists $\gamma \in \mathcal{P}$ such that $OA \in \mathfrak{X}_{\gamma}(O)$ and $\gamma \perp \alpha$. But then we obtain $O = \alpha\beta = \alpha\gamma$ so $\beta = \gamma$. Hence, $\mathfrak{X}_{\alpha}(O)^{\perp} = \mathfrak{X}_{\beta}(O)$ and $\mathfrak{X}_{\alpha}(O) = \mathfrak{X}_{\beta}(O)^{\perp}$. ■

An immediate consequence of Theorem 3.8.18 is the following. For each $\alpha \in \mathcal{P}$ with $O|\alpha$, there exists a unique $\beta \in \mathcal{P}$ such that $O|\beta$, $\mathfrak{X}_{\alpha}(O)^{\perp} = \mathfrak{X}_{\beta}(O)$, and $(\mathfrak{X}_{\alpha}(O)^{\perp})^{\perp} = \mathfrak{X}_{\alpha}(O)$.

Theorem 3.8.19. *If $\hat{a}(O)$ is nonisotropic then there is a unique hyperplane $\mathcal{A}(O)$ such that $\hat{a}(O)^{\perp} = \mathcal{A}(O)$ and $(\hat{a}(O)^{\perp})^{\perp} = \hat{a}(O)$.*

Proof: Let $\alpha \in \mathcal{P}$ such that $a \subset \mathfrak{X}_{\alpha}$, where a is the nonisotropic line associated with $\hat{a}(O) = \{OA : A \in a\}$. Then for $O = \alpha\beta$ we have $a \perp \beta$ and $a \perp g$ for every $g \subset \mathfrak{X}_{\beta}$ with $O \in g$. Thus, $\mathfrak{X}_{\beta}(O) \subset \hat{a}(O)^{\perp}$. Now a is nonisotropic, so by 3.3.8, there exists a unique $h \subset \mathfrak{X}_{\alpha}$ such that $O \in h$ and $a \perp h$. By Axiom U, $\langle \hat{h}(O), \mathfrak{X}_{\beta}(O) \rangle \subset \hat{a}(O)^{\perp}$.

Suppose $1_{\mathfrak{G}} \neq OB \in \hat{a}(O) \cap \langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle$. Then $B \in a \subset \mathfrak{X}_\alpha$ and there exists $OC \in \hat{h}(O)$ and there .exists $OD \in \hat{\mathfrak{X}}_\beta(O)$ such that $OB = OCOD$; that is, there exists $C \in h$ and $D \mid \beta$ such that $B = COD$. because $h \subset \mathfrak{X}_\alpha$, $C|\alpha$ and $C = \alpha\alpha_1$. Because $D|\beta$, $D = \beta\beta_1$ for some $\beta_1 \in \mathcal{P}$. Now $\alpha \mathbin{\underline{\perp}} \beta$ and $\beta \mathbin{\underline{\perp}} \beta_1$, so $\alpha \parallel \beta_1$. But $B = COD$ and $COD = \alpha_1\alpha\alpha\beta\beta\beta_1 = \alpha_1\beta_1$ so that $B|\beta_1, \alpha$ with $\alpha \parallel \beta_1$. This implies that $\alpha = \beta_1$ so $D = O$, $B = C$, and $a = h$, a contradiction. Hence, $\hat{a}(O) \cap \langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle = \{1_{\mathfrak{G}}\}$ and $\mathcal{V} = \hat{a}(O) \oplus \langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle$. If $O \in d$ and $d \perp a$ then $\hat{d}(O) \subset \langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle$. For otherwise we would have

$$\mathcal{V} = \hat{a}(O) \oplus \hat{d}(O) \oplus \langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle,$$

which is not possible. Hence, $\hat{a}(O)^\perp = \langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle$.

On the other hand, from $\hat{\mathfrak{X}}_\beta(O)^\perp = \hat{\mathfrak{X}}_\alpha(O)$, $a \perp h$, and $a \perp \beta$, then, $\hat{a}(O) \subset \langle \hat{h}(O), \hat{\mathfrak{X}}_\alpha(O)\rangle^\perp$. If $O \in g \perp \beta$, then by definition $g \subset \mathfrak{X}_\alpha$. If $O \in g \perp h$, then $g = a$. Hence, $\hat{a}(O) = \langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle^\perp$.

Claim. $\hat{a}(O)^\perp$ is independent of the plane containing a. Suppose $a \subset \mathfrak{X}_\gamma \neq \mathfrak{X}_\alpha$. Then for $O = \gamma\delta$, we have $\gamma \neq \alpha$, $\delta \neq \beta$, and $a \perp \mathfrak{X}_\delta$. Again there exists a unique $l \subset \mathfrak{X}_\gamma$ such that $O \in l$ and $l \perp a$, so that $\hat{a}(O)^\perp = \langle \hat{l}(O), \hat{\mathfrak{X}}_\delta(O)\rangle$ as above. Now any point $P \in \mathfrak{X}$ may be written as $P = P_1OP_2$ with $P_1|\alpha$ and $P_2|\beta$. Since $h \perp a$, then we may write $P_1 = HOA$ with $H \in h$ and $A \in a$. Then

$$\langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle \oplus \hat{a}(O) = \mathcal{V} = \langle \hat{l}(O), \hat{\mathfrak{X}}_\gamma(O)\rangle \oplus \hat{a}(O),$$

and it follows that $\langle \hat{h}(O), \hat{\mathfrak{X}}_\beta(O)\rangle = \langle \hat{l}(O), \hat{\mathfrak{X}}_\gamma(O)\rangle = \mathcal{A}(O)$. ■

Theorem 3.8.20. *If $\hat{a}(O)$ is isotropic then there is an unique hyperplane $\mathcal{A}(O)$ such that $\hat{a}(O)^\perp = \mathcal{A}(O), \hat{a}(O) \subset \mathcal{A}(O)$, and $(\hat{a}(O)^\perp)^\perp = \hat{a}(O)$.*

Proof: Let $\hat{a}(O) \subset \hat{\mathfrak{X}}_\alpha(O)$ and a the isotropic line in $\mathfrak{X}_\alpha$ corresponding to $\hat{a}(O)$. because a is isotropic then $\alpha \in \mathcal{M}$. Let $O = \alpha\beta$, $\beta \in \mathcal{G}$, and put $\mathcal{A}(O) = \langle \hat{a}(O), \hat{\mathfrak{X}}_\beta(O)\rangle$. Since $a \subset \mathfrak{X}_\alpha$, $\alpha \mathbin{\underline{\perp}} \beta$, and $O = \alpha\beta \in a$, then $a \perp \mathfrak{X}_\beta$. Because

a is isotropic then $a \perp a$. By Axiom U and Lemma 3.8.14 it follows that $\langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle = \mathcal{A}(O) \subset \hat{a}(O)^\perp$. If $\hat{g}(O) \subset \langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle$ is isotropic and $g \neq a$, then there exists a unique $\gamma \in \mathcal{M}$ such that $g, a \subset \mathfrak{X}_\gamma$. Because $\hat{g}(O) \subset \mathcal{A}(O) \subset \hat{a}(O)^\perp$, then $g \perp a$, which contradicts Lemma 3.8.11. Thus, $\mathcal{A}(O)$ contains no other isotropic line.

Let $h \perp a$ with $O \in h$. For each $H \in h$ we may write $H = H'OB$, with $H'|\alpha$ and $B|\beta$. If $O \neq A \in a$, it follows that $OB \perp OA$; $OH \perp OA$, so that $OB^O \perp OA$ and $OH' = OHOB^O \perp OA$. This implies that $g_{OH'} = a$ so $H' \in a$ and $\hat{h}(O) \in \langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle$. Thus, $\hat{a}(O)^\perp = \langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle$.

If $\hat{h}(O) \subset \langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle^\perp$, then

$$\hat{h}(O) \subset \mathfrak{X}_\beta(O)^\perp = \mathfrak{X}_\alpha(O) \quad \text{and} \quad \hat{h}(O) \perp \hat{a}(O).$$

Thus, $\hat{h}(O) = \hat{a}(O)$. Hence, $\langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle^\perp = \hat{a}(O)$ and $(\hat{a}(O)^\perp)^\perp = \hat{a}(O)$.

The subspace $\hat{a}(O)^\perp$ is independent of the plane containing a. Suppose $a \subset \mathfrak{X}_\gamma$, $\gamma \in \mathcal{M}$, $\gamma \neq \alpha$. Put $O = \gamma\delta$. Let $\hat{h}(O) \subset \langle \hat{a}(O), \mathfrak{X}_\delta(O) \rangle$. Then $h \perp a$ and $\hat{h}(O) \subset \langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle$. If

$$\hat{k}(O) \subset \langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle$$

then $k \perp a$ and $\hat{k}(O) \subset \langle \hat{a}(O), \mathfrak{X}_\delta(O) \rangle$. Therefore,

$$\langle \hat{a}(O), \mathfrak{X}_\delta(O) \rangle = \langle \hat{a}(O), \mathfrak{X}_\beta(O) \rangle. \blacksquare$$

Remark. If g is a line then g is either nonisotropic or isotropic. From Theorem 3.8.19 and Theorem 3.8.20, if $\mathcal{U} \leq \mathcal{V}$ is a one-dimensional subspace then $\mathcal{U}^\perp$ is a uniquely determined hyperplane.

In an affine space a plane is uniquely determined by two distinct intersecting lines. Let g and h be two distinct intersecting lines and let $\langle g, h \rangle$ denote the unique plane determined by g and h.

Definition 3.8.21. If $\alpha \in \mathcal{P}$ such that $g,h \subset \mathfrak{X}_\alpha$ then we say that $\langle g,h\rangle = \mathfrak{X}_\alpha$ is a *nonsingular plane*. If there does not exist $\alpha \in \mathcal{P}$ such that $g,h \subset \mathfrak{X}_\alpha$ then we say that $\langle g,h\rangle$ is *singular or* $\langle g,h\rangle$ *is a singular plane.*

It is clear that every plane in $(\mathfrak{X},\mathcal{V},\mathcal{K})$ is either singular or nonsingular. Note that by Theorem 3.8.18, if $\mathcal{U} \leq \mathcal{V}$ is a nonsingular two-dimensional subspace of $\mathcal{V}$, then $\mathcal{U}^\perp$ is a uniquely determined nonsingular two-dimensional subspace of $\mathcal{V}$. Now consider the following cases for two distinct intersecting lines g and h.

Suppose that g is isotropic and h is isotropic. Then by Axiom T, there exists a unique $\alpha \in \mathcal{M}$ such that $g,h \subset \mathfrak{X}_\alpha$. If one of g or h is timelike, then by Axiom T, there exists a unique $\alpha \in \mathcal{M}$ such that $g,h \subset \mathfrak{X}_\alpha$.

Thus, if $\langle g,h\rangle$ is singular then either g is isotropic and h is spacelike, or g and h are both spacelike.

Proposition 3.8.22. *Let* $\langle g,h\rangle$ *be a singular plane with* g *isotropic and* h *spacelike. Then* $g \perp h$.

Proof: Let $\{P\} = g \cap h$. By Axiom S2, either $g \perp h$ or for each A in g, there exists B in h such that P and APB are unjoinable. Suppose that $A \in g$, $B \in h$, and P and APB are unjoinable. It must be the case that A is joinable with APB; that is, $g_{A,APB}$ is nonisotropic. Let $g_{PA} = [\delta,\varepsilon]$, so $A,P|\delta,\varepsilon$. If A is unjoinable with APB then A, P, and APB are pairwise unjoinable points, so by Axiom 10, $APB|\delta,\varepsilon$ and $APB \in g$. Thus, $g_{P,APB} = g_{PA} = g$ and $B = PAAPB \in g$, so $g = h$.

Thus, $g_{P,APB} \neq g$ and $g_{P,APB}$ is isotropic, so by Axiom T, there exists a unique $\gamma \in \mathcal{M}$ such that $g, g_{P,APB} \subset \mathfrak{X}_\gamma$. This means that $P,A,APB|\gamma$ so $B = PAAPB|\gamma$, and $h \subset \mathfrak{X}_\gamma$, which contradicts our initial assumption. Therefore, $g \perp h$. ■

Theorem 3.8.23. *Let* $\langle \hat{g}(O),\hat{h}(O)\rangle$ *be a two dimensional singular subspace of* $\mathcal{V}$ *where* $\hat{g}(O)$ *is isotropic and* $\hat{h}(O)$ *is spacelike. Then* $\langle \hat{g}(O),\hat{h}(O)\rangle^\perp$ *is a two dimensional singular subspace of* $\mathcal{V}$ *which contains* $\hat{g}(O)$. *Moreover,* $(\langle \hat{g}(O),\hat{h}(O)\rangle^\perp)^\perp = \langle \hat{g}(O),\hat{h}(O)\rangle$.

Proof: First we observe that if $\mathcal{W}$ is a subspace of $\mathcal{V}$ generated by subspaces $\mathcal{U}$ and $\mathcal{U}'$

then $\mathcal{W}^{\perp} = \langle \mathcal{U},\mathcal{U}' \rangle^{\perp} = \mathcal{U}^{\perp} \cap \mathcal{U}'^{\perp}$, $\mathrm{v} \in \langle \mathcal{U},\mathcal{U}' \rangle^{\perp} \leftrightarrow \mathrm{v} \in \mathcal{U}^{\perp}$, and $\mathrm{v} \in \mathcal{U}'^{\perp} \leftrightarrow \mathrm{v} \in \mathcal{U}'^{\perp} \cap \mathcal{U}^{\perp}$. Consider $\langle \hat{g}(O), \hat{h}(O) \rangle^{\perp} = \hat{g}(O)^{\perp} \cap \hat{h}(O)^{\perp}$. Because $g \perp h$ by Proposition 3.8.22 then there exists $\gamma, \delta \in \mathcal{P}$ such that $O = \gamma\delta$ with $g \subset \mathfrak{X}_{\gamma}$ and $h \subset \mathfrak{X}_{\delta}$. From Theorem 3.8.20, $\hat{g}(O)^{\perp} = \langle \hat{g}(O), \hat{\mathfrak{X}}_{\delta}(O) \rangle$ and by Theorem 3.8.19, $\hat{h}(O)^{\perp} = \langle \hat{l}(O), \hat{\mathfrak{X}}_{\gamma}(O) \rangle$ where $l \subset \mathfrak{X}_{\delta}$ is the unique line through O in $\mathfrak{X}_{\delta}$ orthogonal to h. Hence,

$$\begin{aligned}
\langle \hat{g}(O), \hat{h}(O) \rangle^{\perp} &= \hat{g}(O)^{\perp} \cap \hat{h}(O)^{\perp} \\
&= \langle \hat{g}(O), \hat{\mathfrak{X}}_{\delta}(O) \rangle \cap \langle \hat{l}(O), \hat{\mathfrak{X}}_{\gamma}(O) \rangle \\
&= \langle \hat{g}(O), \hat{l}(O) \rangle.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\langle \hat{g}(O), \hat{l}(O) \rangle^{\perp} &= \hat{g}(O)^{\perp} \cap \hat{l}(O)^{\perp} \\
&= \langle \hat{g}(O), \hat{\mathfrak{X}}_{\delta}(O) \rangle \cap \langle \hat{h}(O), \hat{\mathfrak{X}}_{\gamma}(O) \rangle \\
&= \langle \hat{g}(O), \hat{h}(O) \rangle.
\end{aligned}$$

Now if $g, l \subset \mathfrak{X}_{\varepsilon}$ for some $\varepsilon \in \mathcal{P}$ then by Theorem 3.8.18, for $O = \varepsilon\beta$, $\langle \hat{g}(O), \hat{l}(O) \rangle = \hat{\mathfrak{X}}_{\varepsilon}(O)$ and $\langle \hat{g}(O), \hat{h}(O) \rangle = \langle \hat{g}(O), \hat{l}(O) \rangle^{\perp} = \hat{\mathfrak{X}}_{\varepsilon}(O)^{\perp} = \hat{\mathfrak{X}}_{\beta}(O)$. This says that $g, h \subset \mathfrak{X}_{\beta}$; that is, $\langle g, h \rangle$ is nonsingular. Thus, $\langle \hat{g}(O), \hat{l}(O) \rangle$ is nonsingular.

If $\langle g, h \rangle$ is a singular plane, $O \in g \cap h$, and g and h are spacelike, then by Axiom S1, g is not orthogonal to h. So by Axiom 2, for all $A \in g$, there is a $B \in h$ such that O and AOB are unjoinable. ■

Theorem 3.8.24. *For the above setup:*

1. $g_{O,AOB} \in \langle g, h \rangle$.
2. $g_{O,AOB} \perp g, h$.
3. *if* $C \in g$ *and* $D \in h$ *such that* $g_{O,COD}$ *is isotropic then* $g_{O,COD} = g_{O,AOB}$.
4. $\langle \hat{g}(O), \hat{h}(O) \rangle^{\perp} = \langle \hat{g}_{O,AOB}(O), \hat{g}(O) \rangle^{\perp}$.

Proof: For 1., if there exists $\gamma \in \mathcal{P}$ such that $g_{O,AOB}, g \subset \mathfrak{X}_{\gamma}$, then $O, A, AOB \| \gamma$ implies that $B = OAAO|\gamma$ and $h \subset \mathfrak{X}_{\gamma}$. This yields $\langle g, h \rangle = \mathfrak{X}_{\gamma}$ is nonsingular. For 2.

$g_{O,AOB} \perp g, h$ by Axiom S2 because by 1. above $g_{O,COD} \subset \langle g,h\rangle$ and $\langle g,h\rangle$ is singular. If $C \in g$ and $D \in h$ such that $g_{O,COD}$ is isotropic then by 1,.$g_{O,COD}, g_{O,AOB} \subset \langle g,h\rangle$. If $g_{O,COD} \neq g_{O,AOB}$ then by Axiom T there exists a unique $\eta \in \mathcal{P}$ such that

$$g_{O,COD},\ g_{O,AOB} \subset \mathfrak{X}_{\gamma}.$$

Because two intersecting lines uniquely determine a plane, then $\langle g,h\rangle = \mathfrak{X}_{\gamma}$. Hence, $g_{O,COD} = g_{O,AOB}$. The last conclusion follows from $\langle g,h\rangle = \langle g, g_{O,AOB}\rangle$. ■

If $\langle g,h\rangle$ is a singular plane then from the proof of Proposition 3.8.22 and from Theorem 3.8.24, it follows that for each $P \in \langle g,h\rangle$ there exists a unique isotropic line $l \subset \langle g,h\rangle$ such that $P \in l$; that is, $\langle \hat{g}(O), \hat{h}(O)\rangle$ contains one isotropic line.

To complete the classification of orthogonal subspaces of $\mathcal{V}$, it remains to consider hyperplanes. Now if $(\mathfrak{X}, \mathcal{V}, \mathcal{K})$ is any four dimensional affine space then a line and a plane which intersect in a point uniquely determine a hyperplane and any hyperplane in the space can be characterized as the subspace generated by a corresponding line and plane.

Let $\mathcal{A} = \langle h, \mathfrak{p}\rangle$ be the hyperplane generated by a line h and a plane $\mathfrak{p}$ which intersect in a point O. Then

$$\mathcal{A}(O)^{\perp} = \langle \hat{h}(O), \hat{\mathfrak{p}}(O)\rangle^{\perp} = \hat{h}(O)^{\perp} \cap \hat{\mathfrak{p}}(O)^{\perp}.$$

From Theorem 3.8.19 and Theorem 3.8.20, we know that the dimension of $\hat{h}(O)^{\perp}$ is three. By Theorems 3.8.18, 3.8.23, and 3.8.24 it follows that the dimension of $(\hat{\mathfrak{p}}(O)^{\perp}) = 2$. Consider the following possibilities. If $\hat{h}(O)^{\perp} \cap \hat{\mathfrak{p}}(O)^{\perp} = \{1_{\mathfrak{G}}\}$, then $\hat{h}(O)^{\perp} \oplus \hat{\mathfrak{p}}(O)^{\perp} \subseteq \mathcal{V}$ has dimension five whereas the dimension of $\mathcal{V}$ is four. If $\hat{h}(O)^{\perp} \cap \hat{\mathfrak{p}}(O)^{\perp} = \hat{\mathfrak{p}}(O)^{\perp}$, then it follows that $\hat{\mathfrak{p}}(O)^{\perp} \subseteq \hat{h}(O)^{\perp}$; $\hat{h}(O) \perp \hat{\mathfrak{p}}(O)^{\perp}$ and

$$\hat{h}(O) \subseteq (\hat{\mathfrak{p}}(O)^{\perp})^{\perp} = \hat{\mathfrak{p}}(O).$$

Which contradicts the assumption that h and $\mathfrak{p}$ intersect only in O.

Therefore, $\hat{h}(O)^{\perp} \cap \hat{\mathfrak{p}}(O)^{\perp} = \hat{g}(O)$ for some line g containing O. Because g must be either isotropic or nonisotropic, then the following is true.

Theorem 3.8.25. *If $\mathcal{A}(O)$ is a hyperplane the there is a unique line g such that $\mathcal{A}(O)^{\perp} = \hat{g}(O)$ and $(\mathcal{A}(O)^{\perp})^{\perp} = \mathcal{A}(O)$.* ■

3.9 The Polarity

In this section a polarity is defined so that we can obtain the metric through a process given by Baer [3]. For the convenience of the reader, the pertinent definitions and theorems [3] are given below.

Definition 3.9.1. An *autoduality* π of the vector space $\mathcal{V}$ over the field $\mathcal{K}$ is a correspondence with the following properties:

1. Every subspace $\mathcal{U}$ of $\mathcal{V}$ is mapped onto a uniquely determined subspace $\pi(\mathcal{U})$ of $\mathcal{V}$.
2. To every subspace $\mathcal{U}$ of $\mathcal{V}$ there exists one and only one subspace $\mathcal{W}$ of $\mathcal{V}$ such that $\pi(\mathcal{W}) = \mathcal{U}$.
3. For subspaces $\mathcal{U}$ and $\mathcal{W}$ of $\mathcal{V}$, $\mathcal{U} \leq \mathcal{W}$ if, and only if, $\pi(\mathcal{W}) \leq \pi(\mathcal{U})$.

In other words, an autoduality is a one-to-one monotone decreasing mapping of the totality of the subspaces of $\mathcal{V}$ onto the totality of the subspaces of $\mathcal{V}$.

Definition 3.9.2. An autoduality π of the vector space $(\mathcal{V},\mathcal{K})$ of dimension not less than two is called a *polarity*, if $\pi^2 = 1$, the identity.

Definition 3.9.3. A *semibilinear form* over $(\mathcal{V},\mathcal{K})$ is a pair consisting of an anti-automorphism α of the field $\mathcal{K}$ and a function $f(\mathrm{x}, \mathrm{y})$ with the following properties:

(i) $f(\mathrm{x}, \mathrm{y})$ is, for every $\mathrm{x}, \mathrm{y} \in \mathcal{V}$, a uniquely determined number in $\mathcal{K}$.

(ii) $f(\mathrm{a} + \mathrm{b}, \mathrm{c}) = f(\mathrm{a}, \mathrm{c}) + f(\mathrm{b}, \mathrm{c})$ and $f(\mathrm{a}, \mathrm{b} + \mathrm{c}) = f(\mathrm{a}, \mathrm{b}) + f(\mathrm{a}, \mathrm{c})$, for $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathcal{V}$.

(iii) $f(t\mathrm{x}, \mathrm{y}) = tf(\mathrm{x}, \mathrm{y})$ and $f(\mathrm{x}, t\mathrm{y}) = f(\mathrm{x}, \mathrm{y})\alpha(t)$ for $\mathrm{x}, \mathrm{y} \in \mathcal{V}$ and $t \in \mathcal{K}$.

If $\alpha = 1$ then f is called a bilinear form.

Definition 3.9.4. If f is a semibilinear form over $(\mathcal{V},\mathcal{K})$ and if $\mathcal{U}$ is a subset of $\mathcal{V}$, then

$\{x \in \mathcal{V} : f(x, u) = 0 \text{ for every} u \in \mathcal{U}\}$ and $\{x \in \mathcal{V} : f(u, x) = 0 \text{ for every } u \in \mathcal{U}\}$ are subspaces of $\mathcal{V}$. We say that the autoduality π of $(\mathcal{V}, \mathcal{K})$ upon itself is *represented by the semibilinear form* $f(x, y)$ if

$$\pi(\mathcal{U}) = \{x \in \mathcal{V} : f(x, \mathcal{U}) = 0\} \equiv \{x \in \mathcal{V} : f(x, u) = 0 \text{ for every } u \in \mathcal{U}\}.$$

Theorem 3.9.5.[3] *Autodualities of vector spaces of dimension not less than* 3 *are represented by semibilinear forms.* ■

Theorem 3.9.6.[3] *If the semibilinear forms f and g over $(\mathcal{V}, \mathcal{K})$ represent the same autoduality of $\mathcal{V}$, and if $dim(\mathcal{V}) \geq 2$, then there exists a $0 \neq d \in \mathcal{K}$ such that*

$$g(x, y) = f(x, y)d \qquad \textit{for every } x, y \in \mathcal{V}. \ ■$$

Definition 3.9.7. If π is an autoduality of the vector space $(\mathcal{V}, \mathcal{K})$, then a subspace $\mathcal{W}$ of $\mathcal{V}$ is called an N-subspace of $\mathcal{V}$ with respect to π, if $\langle v\rangle \leq \pi(\langle v\rangle)$ for every $v \in \mathcal{W}$. In this case, π is said to be a *null system* on the subspace $\mathcal{W}$ of $\mathcal{V}$.

Theorem 3.9.8.[3] *Suppose that (f, α) represents the autoduality π of $(\mathcal{V}, \mathcal{K})$. Then π is a null system on the subspace $\mathcal{W} \leftrightarrow f(w, w) = 0$ for every $w \in \mathcal{W}$.* ■

Definition 3.9.9. A line $\langle v\rangle$ is called *isotropic* if $\langle v\rangle \leq \pi(\langle v\rangle)$. (So an isotropic line is an N-line.)

Theorem 3.9.10.[3] *If the semibilinear form (f, α) represents a polarity π, and if $f(w, w) = 1$ for some $w \in \mathcal{V}$, then $\alpha^2 = 1$ and*

$$\alpha(f(x, y)) = f(y, x) \qquad \textit{for every } x, y \in \mathcal{V}.$$

In this case we say that f is α –symmetrical or just symmetrical. ■

Theorem 3.9.11.[3] *Suppose that π is an autoduality of the vector space $(\mathcal{V}, \mathcal{K})$ and that $dim(\mathcal{V}) \geq 3$. Then π is a polarity if, and only if, π is either a null system or else π may be represented by a symmetrical semibilinear form (f, α) with involutorial α.* ■

Theorem 3.9.12.[3] *Suppose that the polarity π of the vector space $(\mathcal{V}, \mathcal{K})$ possess*

isotropic lines and the $dim(\mathcal{V}) \geq 3$. *Then* π *may be represented by bilinear forms if, and only if,*

(a) planes containing more than two isotropic lines are N*-planes and*

(b) $\mathcal{K}$ *is commutative.* ■

Theorem 3.9.13. *Suppose that* π *is a polarity of the vector space* $(\mathcal{V},\mathcal{K})$ *such that the conditions of Theorem* 9.12 *are met. Then from Theorem* 3.9.11 *it follows that if* π *is not a null system then* π *may be represented by symmetrical bilinear forms.* ■

Defining the polarity π and obtaining the metric g. Consider $(\mathcal{V},\mathcal{K})$ constructed in this work. If $\mathcal{U}$ is any subspace of $\mathcal{V}$ then by Theorem 3.8.17, $\mathcal{U}^{\perp}$ is also a subspace of $\mathcal{V}$. From the end of section 3.8, if $\mathcal{U}$ is a subspace of $\mathcal{V}$, $\mathcal{U} \neq \{1_{\mathfrak{G}}\}$, and $\mathcal{U} \neq \mathcal{V}$, then $\mathcal{U}^{\perp}$ is a uniquely determined subspace of $\mathcal{V}$. From Lemma 3.8.11 and Definition 3.8.5, $\{1_{\mathfrak{G}}\}^{\perp} = \mathcal{V}$ and $\mathcal{V}^{\perp} = \{1_{\mathfrak{G}}\}$. So we define the mapping π on the subspaces of $\mathcal{V}$ as follows:

Definition 3.9.14. If $\mathcal{U}$ is a subspace of $\mathcal{V}$ then $\pi(\mathcal{U}) \equiv \mathcal{U}^{\perp}$.

From the remarks above and from Section 3.8 it is clear that to every subspace $\mathcal{U}$ of $\mathcal{V}$ there exists one and only one subspace $\mathcal{W}$ of $\mathcal{V}$ such that $\pi(\mathcal{W}) = \mathcal{U}$.

Theorem 3.9.15. *Let* $\mathcal{U}$ *and* $\mathcal{W}$ *be subspaces of* $\mathcal{V}$, *then* $\mathcal{U} \leq \mathcal{W}$ *if, and only if,* $\pi(\mathcal{W}) \leq \pi(\mathcal{U})$.

Proof: Suppose that $\mathcal{U} \leq \mathcal{W}$. If $OV \in \mathcal{W}^{\perp}$ then $OV \perp OW$ for every $OW \in \mathcal{W}$, $\mathcal{U} \subset \mathcal{W}$, so $OV \perp OU$ for every $OU \in \mathcal{U}$ and $OV \in \mathcal{U}^{\perp}$. Thus, $\mathcal{U} \leq \mathcal{W}$ implies that $\pi(\mathcal{W}) = \mathcal{W}^{\perp} \leq \mathcal{U}^{\perp} = \pi(\mathcal{U})$. Conversely, suppose that $\mathcal{W}^{\perp} \leq \mathcal{U}^{\perp}$. By Lemma 3.8.11 and Definition 3.8.5 we have $(\{1_{\mathfrak{G}}\}^{\perp})^{\perp} = \{1_{\mathfrak{G}}\}$ and $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$. From the previous section it follws that, if $\mathcal{U}$ is a nontrivial proper subspace of $\mathcal{V}$ then $(\mathcal{U}^{\perp})^{\perp} = \mathcal{U}$. because $\mathcal{W}^{\perp}$ and $\mathcal{U}^{\perp}$ are subspaces, then from the first part of the proof we have $\mathcal{W}^{\perp} \leq \mathcal{U}^{\perp}$ implies that $\mathcal{U} = (\mathcal{U}^{\perp})^{\perp} \leq (\mathcal{W}^{\perp})^{\perp} = \mathcal{W}$. ■

Corollary 3.9.16. *From Definition* 3.9.14 *and Theorem* 3.9.15 *above it follows that* $\pi : \mathcal{U} \leq \mathcal{V} \mapsto \mathcal{U}^{\perp} \leq \mathcal{V}$ *is an autoduality. Moreover, because* $(\mathcal{U}^{\perp})^{\perp} = \mathcal{U}$ *for every*

subspace $\mathcal{U}$ of $\mathcal{V}$, then π is a polarity.

Lemma 3.9.17. *Let $\xi \in \mathfrak{G}$ and $OA \in \mathcal{V}$. Then $\pi(\mathcal{U}^{\xi}) = \pi(\mathcal{U})^{\xi}$ for every $\xi \in \mathfrak{G}$ and for every subspace $\mathcal{U} \leq \mathcal{V}$. That is, π is invariant under σ_{ξ} for every $\xi \in \mathfrak{G}$.*

Proof: We have $O(OA)^{\xi} = OO^{\xi}A^{\xi} = B \in \mathfrak{X}$, so $(OA)^{\xi} = OB \in \mathcal{V}$. Hence, from Lemma 3.8.6.the result follows. ■

Theorem 3.9.18. *The polarity π is not a null system.*

Proof: Let a be any nonisotropic line with $O \in a$. By Lemma 3.8.7, a is not orthogonal to itself so that $\hat{a}(O) \nleq \hat{a}(O)^{\perp}$ and hence, $\hat{a}(O) \nleq \pi(\hat{a}(O))$. By Definition 3.9.7, π is not a null system. ■

Theorem 3.9.19. *The polarity π defined above may be represented by bilinear forms.*

Proof:. Let g be any isotropic line through O. By Definition 3.8.5, $\hat{g}(O) \leq \hat{g}(O)^{\perp}$ so that $\hat{g}(O) \leq \pi(\hat{g}(O))$ and $\hat{g}(O)$ is an isotropic line in the sense of Definition 3.9.9. Conversely, if h is a line such that $\hat{h}(O) \leq \pi(\hat{h}(O)) = \hat{h}(O)^{\perp}$ then $h \perp h$ and by Lemma 3.8.7, h must be isotropic in the sense of Section 3.2. Thus, $(\mathcal{V}, \mathcal{K})$ possesses isotropic lines. By Proposition 3.7.3, $\dim(\mathcal{V}) = 4 \geq 3$. From Definition 3.8.21 any plane in $(\mathcal{V}, \mathcal{K})$ is singular or nonsingular. Now a singular plane contains only one isotropic line by Theorem 3.8.24. If $\mathcal{U} \leq \mathcal{V}$ is a nonsingular plane then $\mathcal{U} = \hat{\mathfrak{X}}_{\gamma}(O)$ for some $\gamma \in \mathcal{P}$. If $\gamma \in \mathcal{M}$ then $\mathcal{U}$ has precisely two isotropic lines, as was shown in 3.2.2.8. If $\gamma \in \mathcal{G}$, then by Axiom 12 and Section 3.2, $\mathcal{U}$ does not have isotropic lines. Thus, no plane of $\mathcal{V}$ contains more than two isotropic lines. Since the field $\mathcal{K} = \mathbb{R}$ is commutative then by Theorem 3.8.12 the result follows. ■

Theorem 3.9.20. *The polarity π may be represented by symmetrical bilinear forms; that is, there is a symmetrical bilinear form g such that for any subspace $\mathcal{U}$ of $\mathcal{V}$,*

$$\mathcal{U}^{\perp} = \pi(\mathcal{U}) = \{OA \in \mathcal{V} : g(OA, \mathcal{U}) = 0\} \text{ or}$$
$$g(OA, OB) = O = g(OB, OA) \text{ if, and only if, } OA \perp OB \text{ for } OA, OB \in \mathcal{V}.$$

Proof: This follows directly from Theorems 3.9.18, 3.9.19, and 3.9.13. ■

Thus, we have a metric g, a symmetric bilinear form, induced by our polarity, which agrees with our definition of orthogonal vectors, which in turn is induced by and is defined in terms of the commutation relations of the elements of our generating set $\mathcal{G}$.

Lemma 3.9.21. *Let* g *be a symmetric bilinear form representing* π. *Then* g *is nondegenerate.*

Proof: This follows from the fact that $\pi(\mathcal{V}) = \mathcal{V}^{\perp} = \{1_{\mathfrak{G}}\}$. ■

Theorem 3.9.22. *Let* g *be a symmetric bilinear form representing* π. *Then there is a basis of* $\mathcal{V}$ *such that the matrix of* g *with respect to this basis has the form*

$$\begin{pmatrix} C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & -C \end{pmatrix}, \quad 0 \neq C \in \mathcal{K};$$

that is, there is an orthogonal basis of $\mathcal{V}$ *such that the matrix of* g *with respect to this basis has the above form.*

Proof: Let $\alpha \in \mathcal{M}$ such that $\mathcal{K} \subset \mathfrak{X}_{\alpha}$ and let $O = \alpha\beta$. Let $O|\gamma,\delta \in \mathcal{G}$ such that $\alpha = \gamma\delta$ and $\gamma,\delta \perp \beta$. Let $\varepsilon = \gamma\beta \in \mathcal{M}$ and $\eta = \delta\beta \in \mathcal{M}$.Then $\varepsilon\eta = \gamma\beta\beta\delta = \gamma\delta = \alpha$ so $\varepsilon \perp \eta$. Put $x = [\alpha,\gamma,\delta]$, $y = [\beta,\gamma,\varepsilon]$, $z = [\beta,\eta,\delta]$, and $t = [\alpha,\varepsilon,\eta]$.

Claim. x,y,z,t are four mutually orthogonal nonisotropic lines through O. We have $O = \alpha\beta$, $O \in x,t \subset \mathfrak{X}_{\alpha}$ and $O \in y,z \subset \mathfrak{X}_{\beta}$ so that $x,t \perp y,z$. Also, $O = \alpha\beta = \gamma\delta\beta = \gamma\eta = \delta\varepsilon$ so that $\gamma \perp \eta$ and $\delta \perp \varepsilon$. Then

$$O \in x \subset \mathfrak{X}_{\gamma}, O \in t \subset \mathfrak{X}_{\eta}, O \in \mathfrak{X}_{\gamma}, \text{ and } O \in z \subset \mathfrak{X}_{\eta}$$

implies that $x \perp t$ and $y \perp z$.

The construction of the basis. Let $E \in \mathcal{K}$, the multiplicative identity, considered as a point in $\mathfrak{X}_{\alpha}$. Put $T \equiv EOE^{t} \in t$ and $X \equiv EOE^{x} \in x$. Note that because

$O \in x, t \subset \mathfrak{X}_\alpha$ and $x \perp t$ then $\sigma_x\sigma_t = \sigma_O = \sigma_t\sigma_x$ in $\mathfrak{X}_\alpha$. Now $t \subset \mathfrak{X}_\varepsilon$ and $\varepsilon \in \mathcal{M}$, so there exist precisely two isotropic lines, $\mathcal{K}_{1\varepsilon}$ and $\mathcal{K}_{2\varepsilon}$, through O in $\mathfrak{X}_\varepsilon$. Thus, there is an unique $E_\varepsilon \in \mathcal{K}_{1\varepsilon}$ such that $T = E_\varepsilon O E_\varepsilon^t$. Because $y \subset \mathfrak{X}_\varepsilon$ then $Y \equiv E_\varepsilon O E_\varepsilon^y \in y$ and $\sigma_y\sigma_t = \sigma_O = \sigma_t\sigma_y$ in $\mathfrak{X}_\varepsilon$. Similarly, because $t \subset \mathfrak{X}_\eta$; $\eta \in \mathcal{M}$, there are precisely two isotropic lines, $\mathcal{K}_{1\eta}$ and $\mathcal{K}_{2\eta}$, through O in $\mathfrak{X}_\eta$ and there is E_η in $\mathcal{K}_{1\eta}$ such that $T = E_\eta O E_\eta^t$. because $z \subset \mathfrak{X}_\eta$ then $Z \equiv E_\eta O E_\eta^t \in z$ and $\sigma_z\sigma_t = \sigma_O = \sigma_t\sigma_z$ in $\mathfrak{X}_\eta$. We calculate $XOT = (EOE^x)O(EOE^t) = (EOE)O(E^xOE^t) = (EOE)O(E^{Ot}O^tE^t)$

$$= (EOE)O(E^OOE)^t = (EOE)O(OEOOE)^t = (EOE)OO^t$$

$$= (EOE)OO = EOE \in \mathcal{K}.$$

$YOT = (E_\varepsilon OE_\varepsilon^y)O(E_\varepsilon OE_\varepsilon^t) = (E_\varepsilon OE_\varepsilon)O(E_\varepsilon^O OE_\varepsilon)^t = E_\varepsilon OE_\varepsilon \in \mathcal{K}_{1\varepsilon}$.

$ZOT = (E_\eta OE_\eta^z)O(E_\eta OE_\eta^t) = E_\eta OE_\eta \in \mathcal{K}_{1\eta}$. Thus, if we put $\bar{E}_1 = OX$, $\bar{E}_2 = OY$, $\bar{E}_3 = OZ$, and $\bar{E}_4 = OT$ it follows that the set $\{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4\}$ consists of four mutually orthogonal vectors such that $\bar{E}_i + \bar{E}_4$ is isotropic for $i = 1,2,3$. So if g is a symmetric bilinear form representing π then from $\bar{E}_i + \bar{E}_4 \perp \bar{E}_i + \bar{E}_4$ for $i = 1,2,3$ and $\bar{E}_i \perp \bar{E}_j$ for $i \neq j$ it follows that for $i = 1,2,3$,

$$O = g(\bar{E}_i + \bar{E}_4, \bar{E}_i + \bar{E}_4) = g(\bar{E}_i, \bar{E}_i) + 2g(\bar{E}_i, \bar{E}_4) + g(\bar{E}_4, \bar{E}_4) = g(\bar{E}_i, \bar{E}_i) + g(\bar{E}_4, \bar{E}_4).$$

So that $g(\bar{E}_i, \bar{E}_i) = -g(\bar{E}_4, \bar{E}_4) \neq O$, because $\bar{E}_4$ is nonisotropic. Thus, it remains to show that $\{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4\}$ is a basis for $\mathcal{V}$. But this follows from Section 3.7. ■

Theorem 3.9.23. $(\mathcal{V}, \mathcal{K}, g)$ *is a four-dimensional Minkowski vector space. Moreover,* $(\hat{\mathfrak{X}}_\gamma(O), g)$ *is a four-dimensional Minkowski space for every* γ *in* $\mathcal{P}_O$.

Proof: Put $g(\vec{E}_4, \vec{E}_4) = -1$ and $g(\vec{E}_i, \vec{E}_i) = 1$, for $i = 1,2,3$. Minkowski space is the only nonsingular real four-dimensional vector space with metric

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad ■ \qquad (3.24)$$

In the last section of Chapter 3 we show that each λ in $\mathcal{G}$ can be identified with a spacelike plane and each $\hat{\sigma}_\lambda \in \mathcal{G}^*$ with a reflection about a spacelike plane.

3.10 Spacelike Planes and Their Relections

For each $\lambda \in \mathcal{G}$ with $0 \mid \lambda$, define $\sigma_\lambda : \mathcal{V} \mapsto \mathcal{V}$ by $\sigma_\lambda(OA) = (OA)^\lambda = OA^\lambda$, for $OA \in \mathcal{V}$. To extend this definition to any $\lambda \in \mathcal{G}$, we note that if $\lambda \not| O$ and $OA \in \mathcal{V}$, then there exists a unique D in $\mathfrak{X}$ such that $OO^\lambda A^\lambda = D$; that is, there is a unique D in $\mathfrak{X}$ such that $(OA)^\lambda = O^\lambda A^\lambda = OD$. Thus, we define $\hat{\sigma}_\lambda(OA) = OD$, where $OO^\lambda A^\lambda = D$. We note that if $\lambda \mid O$ then $D = A^\lambda$.

First we show that $\hat{\sigma}_\lambda$ is a semilinear automorphism of $\mathcal{V}$ for each $\lambda \in \mathcal{V}$. [23] Now, a function $f : \mathcal{V} \mapsto \mathcal{V}$ is a semilinear automorphism if, and only if, it has the following properties:

1. f is an automorphism of the additive group of $\mathcal{V}$ onto itself.
2. f sends one dimensional subspaces of $\mathcal{V}$ onto one dimensional subspaces (that is, f is a collineation).
3. If A and B are linearly independent vectors of $\mathcal{V}$, the vectors f(A) and f(B) are also linearly independent.

Theorem 3.10.1. *The map $\hat{\sigma}_\lambda$ is an automorphism of the additive group of $\mathcal{V}$ onto itself.*

Proof: Suppose that $\hat{\sigma}_\lambda(OA) = \hat{\sigma}_\lambda(OB)$. Then $\hat{\sigma}_\lambda(OA) = OD$ where $OD = O^\lambda A^\lambda = O^\lambda B^\lambda$. This implies that $A^\lambda = B^\lambda$ or $A = B$. Thus, $OA = OB$ and $\hat{\sigma}_\lambda$ is injective. To show that $\hat{\sigma}_\lambda$ is onto, let $OB \in \mathcal{V}$ and $A = OO^\lambda B^\lambda$. Then $\hat{\sigma}_\lambda(OA) = O^\lambda A^\lambda = O^\lambda(OO^\lambda B^\lambda)^\lambda = O^\lambda O^\lambda OB = OB$. We show $\hat{\sigma}_\lambda$ is additive. Let $OA, OB \in \mathcal{V}$. Let $D = OO^\lambda A^\lambda$, $F = OO^\lambda B^\lambda$. Then

$$\hat{\sigma}_\lambda(OA + OB) = (OAOB)^\lambda = O^\lambda A^\lambda O^\lambda B^\lambda = ODOF = \hat{\sigma}_\lambda(OA) + \hat{\sigma}_\lambda(OB). \quad \blacksquare$$

Lemma 3.10.2. *The map $\hat{\sigma}_\lambda$ sends one dimensional subspaces of $\mathcal{V}$ onto one dimensional subspaces of $\mathcal{V}$.*

Proof: Let $< OA >$ be a one dimensional subspace of $\mathcal{V}$. Then there exists a line $g = [\alpha,\beta,\gamma]$ such that $OB \in < OA >$ if, and only if, $O,B \in g$. Since $O,B \mid \alpha,\beta,\gamma \leftrightarrow O^{\lambda},B^{\lambda} \mid \alpha^{\lambda},\beta^{\lambda},\gamma^{\lambda}$ then $\hat{\sigma}_{\lambda}(< OA >) =< \hat{\sigma}_{\lambda}(OA) >$ is a one dimensional subspace of $\mathcal{V}$. ■

Lemma 3.10.3. *The transformation, $\hat{\sigma}_{\lambda}$, maps linearly independent vectors of $\mathcal{V}$ to linearly independent vectors.*

Proof: The vectors $OA, OB \in \mathcal{V}$ are linearly independent if, and only if, $g_{OA} \neq g_{OB}$ if, and only if, $g_{O^{\lambda}A^{\lambda}} \neq g_{O^{\lambda}B^{\lambda}}$. Hence, $\hat{\sigma}_{\lambda}$ is a semilinear automorphism of $\mathcal{V}$ for each $\lambda \in \mathcal{G}$. ■

Theorem 3.10.4. *The map $\hat{\sigma}_{\lambda} : \mathcal{V} \mapsto \mathcal{V}$ is a linear automorphism of $\mathcal{V}$ onto $\mathcal{V}$.*

Proof: Consider the definition of a semilinear automorphism [23,defn.73.1]. Let $(\mathcal{V},\mathcal{K})$ and $(\mathcal{V}',\mathcal{K}')$ be vector spaces over the division rings $\mathcal{K}$ and $\mathcal{K}'$, respectively. Suppose that $\mu : \mathcal{K} \mapsto \mathcal{K}'$ is an isomorphism from $\mathcal{K}$ onto $\mathcal{K}'$. A map $\lambda : \mathcal{V} \mapsto \mathcal{V}'$ is called semilinear with respect to μ if

1. $\lambda(A + B) = \lambda(A) + \lambda(B)$ for all $A,B \in \mathcal{V}$.
2. $\lambda(tA) = \mu(t)\lambda(A)$ for all $A \in \mathcal{V}$ and $t \in \mathcal{K}$.

The only isomorphism $\mu : \mathbb{R} \mapsto \mathbb{R}$ is the identity. Thus, $\hat{\sigma}_{\lambda}$ is a linear automorphism of $\mathcal{V}$ onto $\mathcal{V}$. ■

Let $(\mathcal{V},\mathcal{K},g)$ be a metric vector space. A *similarity* γ of $\mathcal{V}$ is a linear automorphism of $\mathcal{V}$ for which there exists a nonzero $r \in \mathcal{K}$ such that $g(\gamma A,\gamma B) = rg(A,B)$, for all $A,B \in \mathcal{V}$. If A is nonisotropic, $r = \frac{g(\gamma A,\gamma A)}{g(A,A)}$. The scalar r is called the square ratio of the similarity.

Lemma 3.10.5. [23] *A linear automorphism of a metric vector space $\mathcal{V}$ is a similarity if, and only if, it preserves orthogonality.*

Proof: For all lines h and l in our space, $h \perp l \leftrightarrow h^{\lambda} \perp l^{\lambda}$, hence, $\hat{\sigma}_{\lambda}(OA) \perp \hat{\sigma}_{\lambda}(OB) \leftrightarrow OA \perp OB$. ■

Theorem 3.10.6. *The linear automorphism $\hat{\sigma}_\lambda$ is an isometry.*

Proof: Let $\gamma : \mathcal{V} \mapsto \mathcal{V}$ be a similarity with square ratio $r \neq 0$. Then for $A, B \in \mathcal{V}$ we have $g(\gamma A, \gamma B) = rg(A,B) = rg(\gamma^{-1}(\gamma A), \gamma^{-1}(\gamma B))$ and $g(\gamma^{-1}(\gamma A), \gamma^{-1}(\gamma B)) = \frac{1}{r} g(\gamma A, \gamma B)$. That is, γ^{-1} has square ratio r^{-1}. Now $\hat{\sigma}_\lambda$ is an involution, so that $\hat{\sigma}_\lambda = \hat{\sigma}_\lambda^{-1}$. Hence, if $r \neq 0$ is the square ratio of $\hat{\sigma}_\lambda$ then $r = \frac{1}{r}$ or $r^2 = 1$, so $r = \pm 1$. [23] Because the field $\mathcal{K}$ is isomorphic to $\mathbb{R}$, then $\mathcal{V}$ is not an Artinian space. Thus, for each similarity $\hat{\sigma}_\lambda$ of $\mathcal{V}$ there is an unique $r > 0$ and there is an unique isometry σ such that $\hat{\sigma}_\lambda = M(0,r)$; where $M(0,r)(A) = rA$ for all A in $\mathcal{V}$. The map $\hat{\sigma}_\lambda$ is thus a similarity with square ratio r^2. Because $r > 0$, then from above, $r = 1$ and it follows that $\hat{\sigma}_\lambda$ is an isometry. ■

Proposition 3.10.7. *The isometry $\hat{\sigma}_\lambda$ is a $180°$ rotation; that is, a reflection about a plane (a two-dimensional subspace of $\mathcal{V}$).*

Proof: Let $\lambda \in \mathcal{G}$. Suppose that $O|\lambda$ and put $O = \lambda\alpha$ with $\alpha \in \mathcal{M}$. Then for all $A|\lambda$ and for all $B|\alpha$, $\hat{\sigma}_\lambda(OA) = (OA)^\lambda = O^\lambda A^\lambda = OA$ and

$$\hat{\sigma}_\lambda(OB) = (OB)^\lambda = O^\lambda B^\lambda = OB^{\alpha\lambda} = OB^O = -OB.$$

Since $\mathfrak{X}_\alpha(O) = \mathfrak{X}_\lambda(O)^\perp$ and $\mathcal{V} = \mathfrak{X}_\lambda(O) \oplus \mathfrak{X}_\lambda(O)^\perp$ then $\hat{\sigma}_\lambda = 1_{\mathfrak{X}_\lambda(O)} \oplus -1_{\mathfrak{X}_\lambda(O)}$. Thus, $\hat{\sigma}_\lambda$ is a reflection about the plane $\mathfrak{X}_\lambda(O)$.

Suppose that $O \not| \lambda$. Let $\varepsilon \mid O$ such that $\varepsilon \parallel \lambda$ and let $P \mid \lambda$ be arbitrary. Then if $R \mid \varepsilon$, $POR = Q \mid \lambda$ and $OR = PQ$. Thus,

$$\hat{\sigma}_\lambda(OR) = (OR)^\lambda = (PQ)^\lambda = PQ = OR \quad \text{and} \quad \hat{\sigma}_\lambda\, 1_{\mathfrak{X}_\varepsilon(O)} = 1_{\mathfrak{X}_\varepsilon(O)}.$$

Now let $\beta|O$ such that $\beta \perp \varepsilon$, so $O = \varepsilon\beta$. Let $B \mid \beta$. Now $\beta \perp \lambda$ because $\varepsilon \parallel \lambda$ and $POB = D|\gamma$ where $P = \lambda\gamma$ and $\gamma \parallel \beta$. We calculate,

$$\hat{\sigma}_\lambda(OB) = (OB)^\lambda = (PD)^\lambda = (PD)^{\gamma\lambda} = (PD)^P = DP = BO = OB^O = -OB.$$

Hence, $\hat{\sigma}_\lambda = 1_{\mathfrak{X}_\varepsilon(O)} \oplus -1_{\mathfrak{X}_\varepsilon(O)^\perp}$ and $\hat{\sigma}_\lambda$ is a reflection about a plane. ■

To show that $\hat{\sigma}_\lambda$ is a reflection about a spacelike plane (Euclidean plane) for every $\lambda \in \mathcal{G}$, we use the following theorem from Snapper and Troyer [23].

Theorem 3.10.8. [23] *Let $(\mathcal{V}, \mathcal{K}, g)$ be an n-dimensional metric vector space over a field $\mathcal{K}$ with metric g, $\mathcal{K} = \mathbb{R}$, and $n = 2$. Every nonsingular real plane has a coordinate system such that the matrix of its metric is one of the following.*

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ - *the Euclidean plane,* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ - *the Lorentz plane, and*

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ - *the negative Euclidean plane.* ■

Hence, the Euclidean plane, the Lorentz plane, and the negative Euclidean plane are the only nonisometric, nonsingular real planes. This also follows from Sylvester's Theorem [23] (It states that there are precisely n+1 nonisometric, nonsingular spaces of dimension n).

Lemma 3.10.9. *Let $\mathcal{K} = \mathbb{R}$, $n = 4$, and $\mathcal{V}$ be Minkowski space. Then the orthogonal complement of a Lorentz plane is a Euclidean plane.*

Proof: Let $\{e_i\}_{i=1,\ldots,4}$ be a basis for $\mathcal{V}$ such that the metric of $\mathcal{V}$ with respect to this basis has matrix of the form (3.24). Let $m = e_3 + e_4$ and $m = e_3 - e_4$ so that $< m >, < n >$ are the unique isotropic lines in the plane $< e_3, e_4 >$. Let α be a Lorentz plane with isotropic basis m_1 and n_1. Then there is an isometry $\sigma :< e_3, e_4 > \rightarrowtail \alpha$ such that $\sigma(m) = m_1$ and $\sigma(n) = n_1$. By the Witt Theorem [23] σ can be extended to an isometry of $\mathcal{V}$, which we also denote by σ. This implies that

$$\sigma : \mathcal{V} =< e_3, e_4 > \oplus < e_1, e_2 > \rightarrowtail \mathcal{V} = \alpha \oplus < \sigma(e_1), \sigma(e_2) >;$$

that is, $\{\sigma(e_1), \sigma(e_2)\}$ is a basis for $\alpha^\perp$. Now σ is an isometry and $g(e_1, e_1) = g(e_2, e_2) = 1$, so the metric g with respect to $\alpha^\perp$ has matrix $\mathbb{I}_2$, the 2x2 identity matrix, and $\alpha^\perp$ is a Euclidean plane. ■

Theorem 3.10.10. *The map $\hat{\sigma}_\lambda$ is a reflection about a spacelike (Euclidean) plane for every $\lambda \in \mathcal{G}$.*

Proof: From the proof of Proposition 3.10.7 it suffices to consider $\lambda \in \mathcal{G}$ with $O|\lambda$. Let $O = \theta\lambda$ with $\theta \in \mathcal{M}$ and let $Q(\cdot)$ denote the quadratic form associated to the metric g obtained in Section 3.9. Because $\theta \in \mathcal{M}$ there are precisely two isotropic lines $\mathcal{K}_{\theta_1}$, $\mathcal{K}_{\theta_2} \subset \mathfrak{X}_\theta$. Let $O \neq A \in \mathcal{K}_{\theta_1}$ and $O \neq B \in \mathcal{K}_{\theta_2}$. Then OA and OB are isotropic vectors which form a basis for $\hat{\mathfrak{X}}_\theta(O)$ and $Q(OA) = 0 = Q(OB)$. Therefore the metric of $\hat{\mathfrak{X}}_\theta(O)$ with respect to OA and OB has matrix

$$g' = \begin{pmatrix} OA \\ OB \end{pmatrix} \begin{pmatrix} OA & OB \end{pmatrix} = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix};$$

where the products of the matrix elements are the inner products defined by the metric g and $g(OA, OB) = r \neq 0$. Since $\frac{1}{r} \cdot OA \in \mathcal{K}_{\theta_1}$, and $OB \in \mathcal{K}_{\theta_2}$ then $\mathcal{K}_{\theta_1}$, $\mathcal{K}_{\theta_2}$ also form an isotropic basis for $\hat{\mathfrak{X}}_\theta(O)$. Hence we may assume that $r = 1$. Let

$$OT_1 = \frac{1}{\sqrt{2}}(OA - OB) \quad \text{and} \quad OX_1 = \frac{1}{\sqrt{2}}(OA + OB).$$

Then $g'(OX_1, OT_1) = 0$, $Q'(OX_1) = 1$, and $Q'(OT_1) = -1$. Because $OX_1 \perp OT_1$ and $\hat{\mathfrak{X}}_\theta(O)$ is nonsingular then OX_1 and OT_1 are linearly independent and hence, form a basis for $\hat{\mathfrak{X}}_\theta(O)$. Moreover, the metric of $\hat{\mathfrak{X}}_\theta(O)$ with respect to $OX_1$and OT_1 has the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore, by Theorem 3.10.8, $\hat{\mathfrak{X}}_\theta(O)$ is a Lorentz plane and by Lemma 3.10.9, $\hat{\mathfrak{X}}_\lambda(O) = \hat{\mathfrak{X}}_\theta(O)^\perp$ is a Euclidean plane. ■

CHAPTER 4
AN EXAMPLE OF THE THREE-DIMENSIONAL MODEL

This chapter begins by considering a net of von Neumann algebras, $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in I}$, and a state ω, coming from a finite component Wightman quantum field theory in three-dimensional Minkowski space. There are various senses to the phrase "coming from a Wightman quantum field theory". The assumption here is the version given by Bisognano and Wichmann [5]. That is, given a finite component Wightman quantum field, $\phi(x)$, assume that the quantum field operator, $\phi(f)$, is essentially self-adjoint and its closure is affiliated with the algebra $\mathcal{R}(\mathcal{O})$ (in the sense of von Neumann algebras) for every test function f whose support lies in the spacetime region $\mathcal{O}$. Driessler, Summers, and Wichmann show these conditions can be weakened [15]. But free boson field theories satisfy these conditions in three-dimensional Minkowski space [5].

For such theories the modular involutions, $J_{\mathcal{O}}$, associated by Tomita-Takesaki theory to the vacuum state and local algebras of wedgelike regions, $\mathcal{O}$, in three-dimensional Minkowski space, act like reflections about the spacelike edge of the wedge [5]. Since the modular involutions have that action upon the net, the hypotheses of Buchholz, Dreyer, Florig, and Summers (BDFS) are satisfied [6]. Therefore the Condition of Geometric Modular Action, CGMA, obtains for the set of wedgelike regions [27] in Minkowski space. The precise wording of this version of the CGMA is given below.

Let l_1 and l_2 be two lightlike linearly independent vectors belonging to the forward light cone in three-dimensional Minkowski space. The wedges are defined as the subsets $W[l_1,l_2] \equiv \{\alpha l_1 + \beta l_2 + l^{\perp} \in \mathbb{R}^{1,2} : \alpha > 0,\ \beta < 0,\ (l^{\perp},l_i) = 0,\ i = 1,2\}$, where $(\ ,\ )$ denotes the Minkowski inner product.

Let $l_1 = (1,1,0)$ and $l_2 = (1,-1,0)$ be lightlike vectors in $\mathbb{R}^{1,2}$, three-dimensional Minkowski space, and let $\mathcal{P}$ be the Poincaré group, the isometry group of this space. Then the set of wedges, $\mathcal{W}$, is given by $\mathcal{W} = \{\lambda W[l_1,l_2] : \lambda \in \mathcal{P}\}$, where

$$\lambda W[l_1,l_2] = \{\lambda(x) : x \in W[l_1 l_2]\}.$$

The CGMA for Minkowski space is defined as follows. Let $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be a net of von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with common cyclic and separating vector $\Omega \in \mathcal{H}$, satisfying the abstract version of the CGMA and where the index set I is chosen to be the collection of wedgelike regions $\mathcal{W}$ in $\mathbb{R}^{1,2}$ defined as above. Recall from Chapter 1, with $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \mathcal{H}, \Omega)$ there is the following.

1. A collection of modular involutions $\{J_W\}_{W\in\mathcal{W}}$.
2. The group $\mathcal{J}$ generated by $\{J_W\}_{W\in\mathcal{W}}$.
3. A collection of involutory transformations on $\mathcal{W}$, $\{\tau_W\}_{W\in\mathcal{W}}$.
4. The group $\mathcal{T}$ generated by $\{\tau_W\}_{W\in\mathcal{W}}$.

Assume also that:

5. The group $\mathcal{T}$ acts transitively upon the set $\mathcal{W}$, that is, for every $W_1, W_2 \in \mathcal{W}$ there is a $W_3 \in \mathcal{W}$ such that $\tau_{W_3}(W_1) = W_2$.

Note that this assumption is implied by the algebraic condition that the set $\{adJ_W\}_{W\in\mathcal{W}}$ acts transitively upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. At this point the following two assumptions are added ([27] which have been verified for general Wightman fields).

4.6 For $W_1, W_2 \in \mathcal{W}$, if $W_1 \cap W_2 \neq \varnothing$, then Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$.

4.7 For $W_1, W_2 \in \mathcal{W}$, if Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$, then $\overline{W}_1 \cap \overline{W}_2 \neq \varnothing$.

The CGMA for Minkowski space is the abstract version of the CGMA with the choice of $\mathcal{W}$ for the index set I, together with assumptions 4.6 and 4.7 above less the transitivity assumption [6].

Buchholz, Dreyer, Florig, and Summers [6] showed that with the above assumptions one can construct a subgroup $\mathfrak{P}$ of the Poincaré group $\mathcal{P}$, which is isomorphic to $\mathcal{T}$ and related to the group $\mathcal{T}$ in the following way. For each $\tau \in \mathcal{T}$ there exists an element $g_\tau \in \mathfrak{P}$ such that $\tau(W) = g_\tau W \equiv \{g_\tau(x) : x \in W\}$. To each of the defining involutions $\tau_W \in \mathcal{T}$, $W \in \mathcal{W}$, there is a unique corresponding $g_W \in \mathfrak{P} \subset \mathcal{P}$ [6]. Moreover, BDFS obtained the following (suitably modified for three dimensions and abbreviated for our purposes).

Theorem 4.1 [6] *Let the group $\mathcal{T}$ act transitively upon the set $\mathcal{W}$ of wedges in $\mathbb{R}^{1,2}$, and let $\mathfrak{P}$ be the corresponding subgroup of $\mathcal{P}$. Moreover, let g_W be the corresponding involutive element of $\mathcal{P}$ corresponding to the involution $\tau_W \in \mathcal{T}$. Then g_W is a reflection about the spacelike orthogonal line which forms the edge of the wedge W. In particular, one has $g_w W = W,'$ the causal complement of W, for every $W \in \mathcal{W}$. In addition, $\mathfrak{P}$ exactly equals the proper Poincaré group $\mathcal{P}_+$.* ■

Recall from Chapter 2 that the initial model of $(\mathcal{G}, \mathfrak{G})$ is as a group plane. This means that each $g \in \mathcal{G}$ is viewed as a line in a plane and each $P = gh$, $g|h$, as a point in a plane. Let us call the axiom system given in Chapter 2 $as - \mathcal{A}$. Thus $as - \mathcal{A}$ is a set of axioms about "points" and "lines" in a "plane".

Let $\mathbb{P}$ denote the collection of points $P \in \mathfrak{G}$. For each $\alpha \in \mathfrak{G}$ define the map $\sigma_\alpha : \mathbb{P} \cup \mathcal{G} \to \mathbb{P} \cup \mathcal{G}$ by

$$\sigma_\alpha(P) \equiv P^\alpha \equiv \alpha P \alpha^{-1} \quad \text{for } P \in \mathcal{P} \quad \text{and } \sigma_\alpha(g) \equiv g^\alpha \equiv \alpha g \alpha^{-1} \quad \text{for } g \in \mathcal{G}.$$

Since $(\mathcal{G}, \mathfrak{G})$ is an invariant system then each σ_α is a bijective mapping of the set of points and the set of lines, each onto itself, which preserves the incidence and orthogonality relations, defined by "|", of the plane. We say that σ_α is a *motion* of the group plane. Since $\mathcal{G}$ generates $\mathfrak{G}$ then the set of line reflections $\mathcal{G}_\sigma = \{\sigma_g : g \in \mathcal{G}\}$ generates the group of motions $\mathfrak{G}_\sigma = \{\sigma_\alpha : \alpha \in \mathfrak{G}\}$. Let $\Phi : \mathfrak{G} \to \mathfrak{G}_\sigma$ be the map defined by $\Phi(\alpha) = \sigma_\alpha$, for $\alpha \in \mathfrak{G}$. Then Φ is in fact a group isomorphism [2]. This

means that $(\mathcal{G}, \mathfrak{G})$ is isomorphic to $(\mathcal{G}_\sigma, \mathfrak{G}_\sigma)$ (in the sense that $\mathcal{G}$ is equivalent to $\mathcal{G}_\sigma$ as sets and $\Phi(\mathcal{G}) = \mathcal{G}_\sigma$, $\Phi(\mathfrak{G}) = \mathfrak{G}_\sigma$, where Φ is a group isomorphism). This implies that $\Phi(as - \mathcal{A})$, which we denote by $as - \Phi(\mathcal{A})$, is an axiom system concerning the group of motions; line reflections of a plane, the group it generates, and point reflections of a plane. A plane whose points and lines satisfy $as - \mathcal{A}$.

As was shown in Chapter 2, given $(\mathcal{G}, \mathfrak{G})$ satisfying $as - \mathcal{A}$, one obtains $\mathbb{R}^{1,2}$, three-dimensional Minkowski space. Under the identifications given in Chapter 2, we find that each $g \in \mathcal{G}$ corresponds to a spacelike line in $\mathbb{R}^{1,2}$. Thus, $as - \Phi(\mathcal{A})$ is a set of true statements concerning reflections about spacelike lines and the motions such reflections generate in three-dimensional Minkowski space. Moreover, since such motions are in fact isometries in $\mathbb{R}^{1,2}$ then $\Phi(\mathfrak{G})$ is isomorphic to a subgroup of the three-dimensional Poincaré group.

Theorem 4.2 *Under the same conditions as in Theorem* 4.1 *it follows that* $(\{\tau_W\}_{W \in \mathcal{W}}, \mathcal{T})$ *acting on* W *satisfies* $as - \Phi(\mathcal{A})$.

Proof: From Theorem 4.1 we have $(\{g_W\}_{W \in \mathcal{W}}, \mathfrak{P})$ satisfies $as - \Phi(\mathcal{A})$ since $\mathfrak{P}$ is the subgroup of $\mathcal{P}$ generated by reflections about spacelike lines. Also from Theorem 4.1, $(\{\tau_W\}_{W \in \mathcal{W}}, \mathcal{T})$ is isomorphic to $(\{g_W\}_{W \in \mathcal{W}}, \mathfrak{P})$ so $(\{\tau_W\}_{W \in \mathcal{W}}, \mathcal{T})$ satisfies $as - \Phi(\mathcal{A})$. ■

The net continuity condition assumed by BDFS [6] for the next theorem was later shown to be superfluous [8] for this theorem and the remaining theorems.

Theorem 4.3 [6] *Assume the CGMA with the spacetime* $\mathbb{R}^{1,2}$ *and* $\mathcal{W}$ *the described set of wedges. If* $\mathcal{J}$ *acts transitively upon the set* $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ *then there exists a strongly (anti-) continuous unitary representation* $U(\mathcal{P}_+)$ *of the proper Poincaré group which acts geometrically correctly upon the net* $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ *and which satisfies* $U(g_W) = J_W$, *for every* $W \in \mathcal{W}$. *Moreover,* $U(\mathcal{P}_+^\uparrow)$ *equals the subgroup of* $\mathcal{J}$ *consisting of all products of even numbers of* J_W *'s and* $\mathcal{J} = U(\mathcal{P}_+^\uparrow) \cup J_{W_R} U(\mathcal{P}_+^\uparrow)$, *where*
$W_R \equiv \{x \in \mathbb{R}^{1,2} : x_1 > |x_0|\}$. ■

Theorem 4.4 *Under the same hypotheses as Theorem 4.3, the group $\mathcal{J}$ is isomorphic to $\mathcal{P}+ \; = \mathfrak{P}$, which is generated by the set of involutions $\{g_W \mid W \in \mathcal{W}\}$. Moreover, $(\{J_w\}_{W\in\mathcal{W}}, \mathcal{J})$ satisfies $as - \Phi(\mathcal{A})$.*

Proof: By Proposition 1.1 there is surjective homomorphism $\xi : \mathcal{J} \to \mathcal{T}$, where the kernel of ξ, $\ker(\xi)$, is contained in the center of $\mathcal{J}$, $\mathcal{Z}(\mathcal{J})$. By Theorem 4.3 there is a faithful representation $U(\mathcal{P}_+)$ such that $U(g_w) = J_W$, for every $W \in \mathcal{W}$. Since the center of $\mathcal{P}_+^\uparrow$ is trivial, $U(\cdot)$ is a faithful representation of $\mathcal{P}_+^\uparrow$ and hence an injective map preserving the algebraic relations, $\mathcal{Z}(\mathcal{J})$ is trivial. This implies that $\ker(\xi)= \{1\}$ and ξ is an isomorphism. If $\Psi : \mathcal{T} \to \mathfrak{P}$ denotes the isomorphism of $\mathcal{T}$ and $\mathfrak{P}$ given by BDFS from Theorem 4.1 then $\Psi \circ \xi : \mathcal{J} \to \mathfrak{P} = \mathcal{P}_+$ is an isomorphism. It therefore follows that the pair $(\{J_W\}_{W\in\mathcal{W}}, \mathcal{J})$ satisfies $as - \Phi(\mathcal{A})$. ■

We can now give the main result of this chapter.

Theorem 4.6 *Any state and net of von Neumann algebras, coming from a (finite component) Wightman quantum field in three-dimensional Minkowski space, which satisfies the Wightman axioms, provides a set of modular involutions satisfying $as - \Phi(\mathcal{A})$.* ■

As a final remark we note that since free boson field theories satisfy the Wightman axioms and therefore the CGMA for Minkowski space holds, then these theories give a concrete example of the three-dimensional case of this dissertation.

CHAPTER 5
CONCLUSION

At this point it is useful to briefly recall the starting point of this thesis, to restate the problem, and to summarize the results obtained. It is assumed that there is a net of C^*-algebras $\{\mathcal{A}_i\}_{i\in I}$, each of which is a subalgebra of a C^*-algebra $\mathcal{A}$, and a state ω on $\mathcal{A}$. If this net and state satisfy the Condition of Geometric Modular Action, CGMA, is it possible to determine the spacetime symmetries (the isometry group), the dimension of the spacetime, and even the spacetime itself, without any assumption about the dimension or the topology of the underlying spacetime?

Recall also that the resolution of these questions involved two steps. First, given a set of involution elements $\mathcal{G}$ and the group $\mathfrak{G}$ it generates, find necessary conditions on the pair $(\mathcal{G}, \mathfrak{G})$ that will allow a construction of three- and four-dimensional Minkowski space. Moreover, this should be done in such a way that the generating involutions can be identified with spacelike lines or spacelike planes and their respective reflections. This was done for three-dimensional Minkowski space in Chapter 2 and for four-dimensional Minkowski space in Chapter 3.

The second step of this process is to determine what additional structure on the index set I would yield algebraic relations among the modular involutions, $\{J_i\}_{i\in I}$, such that the pair $(\{J_i\}_{i\in I}, \mathcal{J})$ satisfies the axiom systems given in Chapters 2 and 3. Using the work of Buchholz, Dreyer, Florig, and Summers [6] and the work of Wiesbrock [29], we are able to obtain a result concerning the above step in three dimensions. First we briefly recall the abstract version of the Condition of Geometric Modular Action, CGMA, described in Chapter 1. We assume there is a net, $\{\mathcal{R}_i\}_{i\in I}$,

of von Neumann algebras acting on a Hilbert space $\mathcal{H}$, where the index set I is a partially-ordered set. There is a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for each $\mathcal{R}_i$, $i \in I$. From the modular theory of Tomita-Takesaki we then obtain a collection, $\{J_i\}_{i \in I}$, of modular involutions which generates a group $\mathcal{J}$ and a collection, $\{\Delta_i\}_{i \in I}$, of modular operators. The assignment $i \mapsto \mathcal{R}_i$ is an order-preserving bijection and each ad J_i leaves the set $\{\mathcal{R}_i\}_{i \in I}$ invariant. The last two assumptions imply that for each $i \in I$, there is an order-preserving bijection τ_i on I such that $J_i \mathcal{R}_j J_i = \mathcal{R}_{\tau_i(j)}$, $j \in I$. The group generated by $\{\tau_i\}_{i \in I}$ is denoted by $\mathcal{T}$ and forms a subgroup of the transformations on the index set I. Assume also that the two intersection conditions for wedges given in Chapter 4 also hold as part of the CGMA for what follows.

To help explain the additional assumptions used in our result we give the following definitions and theorems.

Definition 5.1 [6] *The Modular Stability Condition* (MSC). Let $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ be a net of von Neumann algebras satisfying the CGMA where the index set I is the set of wedgelike regions $\mathcal{W}$ in $\mathbb{R}^{1,2}$ described in Chapter 4. Then the *modular stability condition* is satisfied if the modular unitaries are contained in the group $\mathcal{J}$ generated by the modular involutions; that is,

$$\Delta_W^{it} \in \mathcal{J} \ \text{ for all } t \in \mathbb{R} \text{ and } W \in \mathcal{W}.$$

Theorem 5.2 [6] *Assume the CGMA for three-dimensional Minkowski space with* (4.6) *and* (4.7), *where the index set* $I = \mathcal{W}$, *the collection of wedgelike regions in* $R^{1,2}$. *Assume also the transitivity of the adjoint action of* $\mathcal{J}$ *on the net* $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. *Let* $U(\mathbb{R}^{1,2})$ *be the representation of the translation group. If* $\Delta_W^{it} \in \mathcal{J}$, *for all* $t \in \mathbb{R}$ *and some* $W \in \mathcal{W}$, *that is, if the modular stability condition obtains, then* $sp(U) \subset \overline{V}_+$ *or* $sp(U) \subset \overline{V}_-$. *Moreover, for every future-directed lightlike vector* ℓ *such that* $W + \ell \subset W$, *there holds the relation* $\Delta_W^{it} U(\ell) \Delta_W^{-it} = U(e^{-\alpha t}\ell)$, *for all* $t \in \mathbb{R}$, *where* $\alpha = \pm 2\pi$.

Definition 5.3 [29] Let $\mathcal{N}$, $\mathcal{M}$ be von Neumann algebras acting on a Hilbert Space $\mathcal{H}$. Let Ω be a common cyclic and separating vector in $\mathcal{H}$. If $\Delta_{\mathcal{M}}^{-it}\,\mathcal{N}\,\Delta_{\mathcal{M}}^{it} \subset \mathcal{N}$, for all $t \geq 0$, we call $(\mathcal{N} \subset \mathcal{M}, \Omega)$ a +-half-sided modular inclusion (+-hsm). If $\Delta_{\mathcal{M}}^{it}\,\mathcal{N}\,\Delta_{\mathcal{M}}^{-it} \subset \mathcal{N}$, for all $t \leq 0$, we call $(\mathcal{N} \subset \mathcal{M}, \Omega)$ a —half-sided modular inclusion (—hsm).

Definition 5.4 [29] Let $\mathcal{N}$, $\mathcal{M}$ be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with $\Omega \in \mathcal{H}$ a common cyclic and separating vector for $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{N} \cap \mathcal{M}$.

1. If $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N}, \Omega)$ and $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{M}, \Omega)$ are $\pm$-hsm inclusions.
2. And if $J_{\mathcal{N}}(s - \lim_{t\to\mp\infty} \Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it})J_{\mathcal{N}} = s - \lim_{t\to\mp\infty} \Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it}$.

Then we say that such a pair $((\mathcal{N}, \mathcal{M}), \Omega)$ has $(\pm)$ modular intersection, $\pm$ mis.

Theorem 5.5 [29] *Let* $\mathcal{N}$, $\mathcal{M}$, $\mathcal{L}$, *and* $\hat{\mathcal{N}}$ *be von Neumann algebras acting on a Hilbert space* $\mathcal{H}$ *with a common cyclic and separating vector* $\Omega \in \mathcal{H}$. *Assume the following.*

I. 1. $(\mathcal{N}, \mathcal{M}, \Omega)$ *is* $-$ *mis,*

2. $(\mathcal{L}, \mathcal{M}, \Omega)$ *is* $+$ *mis,*

3. $(\mathcal{L}, \mathcal{N}', \Omega)$ *is* $-$ *mis,*

II. 1. $(\hat{\mathcal{N}} \subset \mathcal{N}, \Omega)$ *is* $-$ *hsm,*

2. $Ad\, J_{\mathcal{M}}(J_{\hat{\mathcal{N}}}J_{\mathcal{N}}) = J_{\mathcal{N}}J_{\hat{\mathcal{N}}}$,

3. $[Ad\, J_{\mathcal{L}}(J_{\hat{\mathcal{N}}}J_{\mathcal{N}}), J_{\hat{\mathcal{N}}}J_{\mathcal{N}}] = 0$,

III. 1. $Ad\,(Ad\, J_{\mathcal{L}}(J_{\hat{\mathcal{N}}}J_{\mathcal{N}})^4)\,(Ad\,\Delta_{\mathcal{M}}^{-it_0}J_{\mathcal{L}}(J_{\mathcal{N}}J_{\hat{\mathcal{N}}}))(J_{\hat{\mathcal{N}}}J_{\mathcal{N}})^2(\mathcal{N}) \subset \mathcal{N}$,

with $t_0 = \frac{1}{2\pi}\ln 2$.

Then the modular groups

$$\Delta_{\mathcal{N}}^{it},\ \Delta_{\mathcal{M}}^{ir},\ \Delta_{\mathcal{L}}^{is}, \text{and}\ \Delta_{\hat{\mathcal{N}}}^{iv}, \quad \text{for } t, r, s, v \in \mathbb{R},$$

generate a unitary representation of the 2+1*-dimensional Poincaré group.* ■

Remarks [29] The conditions in I. give a unitary representation of the 2+1-dimensional homogenous Lorentz group. The hsm inclusion of condition II. equips us with a representation of the translations along some light ray. The product of the two modular conjugations is then a finite translation of this kind. Moreover, due to

the result of Bisognano and Wichmann [5], the modular conjugations of the wedge algebras act as reflections. These properties are encoded in condition II.2 and II.3.

A physical framework will now be given as a description of quantum field theories in terms of local nets of algebras [17]. The basic assumptions are the following. Let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{D}} \subset \mathcal{B}(\mathcal{H})$ be a net of von Neumann algebras indexed by the closed double cones $\mathcal{D}$ in $\mathbb{R}^{1,2}$ which satisfy the following properties:

1. (Isotony) If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.
2. (Locality) If $\mathcal{O}_1 \subset \mathcal{O}_2'$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$.

Where $\mathcal{A}(\mathcal{O})'$ denotes the commutant of $\mathcal{A}(\mathcal{O})$ in $\mathcal{B}(\mathcal{H})$ and $\mathcal{O}'$ is the causal complement of $\mathcal{O} \subset \mathbb{R}^{1,2}$.

3. (Poincaré covariance) There is a unitary representation $\mathcal{U} : SO^{\uparrow}(2,1) \rhd \mathbb{R}^{1,2} \to \mathcal{U}(\mathcal{H})$ of the Poincaré group with positive energy.
4. (Vacuum vector) There is a unique $\mathcal{U}$–invariant vector $\Omega \in \mathcal{H}$.

The algebra $\mathcal{A}(\mathcal{O})$, the inductive limit of the net, is called the local algebra of observables localized in $\mathcal{O} \subset \mathbb{R}^{1,2}$.

As was mentioned in Chapter 4, if the local net is generated by Wightman fields then the modular groups associated with algebras of observables localized in wedges act as Lorentz boosts in the directions of different wedges and the modular conjugations act as reflections [5].

In particular, the adjoint action of the modular conjugations on the net act as reflections about the spacelike edge of the wedge. For what follows we shall call the properties in the previous paragraph the Bisognano and Wichmann property.

Theorem 5.6 [29] *Let $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^{1,2}$, be a local net fulfilling the Bisognano and Wichmann property for wedges. Let $\mathcal{N} \equiv \mathcal{A}(W[l_1,l_2])$, $\mathcal{M} \equiv \mathcal{A}(W[l_1,l_3])$, $\mathcal{L} \equiv \mathcal{A}(W[l_2,l_3])$, and $\hat{\mathcal{N}} \equiv \mathcal{A}(W[l_1,l_2,-l_1])$, where l_1,l_2, and l_3 are three linearly independent light rays. Then this set of algebras together with the vacuum vector Ω fulfill the assumptions of Theorem* 5.4. *Conversely, let $\mathcal{N}$, $\mathcal{M}$, $\mathcal{L}$, and $\hat{\mathcal{N}}$ be a set of* 4

von Neumann algebras acting on a Hilbert space $\mathcal{H}$ together with a common cyclic and separating vector $\Omega \in \mathcal{H}$, which fulfill conditions I.-III. of Theorem 5.4. Then these data determine a local (Bisognano-Wichmann) net $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$, $\mathcal{O} \subset \mathbb{R}^{1,2}$, such that the incident algebras become the wedge algebras of the constructed net as in the first part. ■

Theorem 5.7 *Let $\{\mathcal{R}_i\}_{i\in I}$ be a net of von Neumann algebras acting on a Hilbert space $\mathcal{H}$, together with a common cyclic and separating vector $\Omega \in \mathcal{H}$ satisfying the CGMA with the MSC. Assume also that there exist $i_{\mathcal{N}}, j_{\mathcal{M}}, k_{\hat{\mathcal{N}}}, l_{\mathcal{L}} \in I$ such that $\mathcal{R}_{i_{\mathcal{N}}}, \mathcal{R}_{j_{\mathcal{M}}}, \mathcal{R}_{k_{\hat{\mathcal{N}}}}$, and $\mathcal{R}_{l_{\mathcal{L}}}$ satisfy the hypotheses of Theorem 5.5. Then the following are true.*

1. *There is an injective map $F : \mathcal{W} \to I$, such that for each $W \in \mathcal{W}$, the modular objects $J_{F(W)}$, $\Delta^{it}_{F(W)}$ have the Bisognano-Wichmann Property when acting upon $\mathcal{R}_{F(W)}$.*
2. *The modular unitaries $\Delta^{it}_{i_{\mathcal{N}}}$, $\Delta^{ir}_{j_{\mathcal{M}}}$, $\Delta^{is}_{k_{\hat{\mathcal{N}}}}$, and $\Delta^{ip}_{l_{\mathcal{L}}}$ generate a continuous unitary representation of $\mathcal{P}^{\uparrow}_{+}$ which acts covariantly upon $\{\mathcal{R}_{F(W)}\}_{W\in\mathcal{W}}$.*

If the additional intersection assumptions, (4.6) *and* (4.7), *are made on the subnet $\{\mathcal{R}_{F(W)}\}_{W\in\mathcal{W}}$, then:*

3. *The CGMA as stated for Minkowski space holds for the subnet $\{\mathcal{R}_{F(W)}\}_{W\in\mathcal{W}}$, as does the MSC.*
4. *The modular conjugations $\{adJ_{F(W)}\}_{W\in\mathcal{W}}$, and thus $(\{\tau_{F(W)}\}_{W\in\mathcal{W}}, \mathcal{T})$, satisfy the axioms of Chapter* 2.
5. *There is a continuous (anti-) unitary representation of $\mathcal{P}_+$ acting covariantly upon $\{\mathcal{R}_{F(W)}\}_{W\in\mathcal{W}}$.*

Proof: For the proof of Teorem 5.7.1, we give the construction Wiesbrock gave in the proof of Theorem 5.6 [29]. Let $l_1 = (1,1,0)$, $l_2 = (1,-1,0)$, $l_3 = (1,0,1) \in \mathbb{R}^{1,2}$. The local algebra of observables to wedges is defined by

$$\mathcal{R}(W[l_1,l_2]) \equiv \mathcal{R}_{i_{\mathcal{N}}}, \quad \mathcal{R}(W[l_1,l_3]) \equiv \mathcal{R}_{j_{\mathcal{M}}}, \quad \mathcal{R}(W[l_2,l_3]) \equiv \mathcal{R}_{l_{\mathcal{L}}}.$$

For arbitrary linearly independent light rays $l_i, l_j \in \mathbb{R}^{1,2}$ pointing to the future, let

$\Lambda_{l_i,l_j} \in SO^{\uparrow}(1,2)$ with $l_i = \Lambda_{l_i,l_j} l_1$, and $l_j = \Lambda_{l_i,l_j} l_2$. This element in $SO^{\uparrow}(1,2)$ is uniquely defined up to a multiplication by a boost of type $\Lambda_{l_1,l_2}(t)$, $t \in \mathbb{R}$ with the given asymptotics l_1, l_2. Let $\mathcal{U}$ denote the unitary representation of the Poincaré group according to Theorem 5.4, that is, let

$$\mathcal{U}(\Lambda_{l_1,l_2}(t)) \equiv \Delta^{it/2\pi}_{\mathcal{R}_{i_\mathcal{N}}}, \quad \mathcal{U}(\Lambda_{l_1 l_2}(t)) \equiv \Delta^{it/2\pi}_{\mathcal{R}_{j_\mathcal{M}}}, \quad \mathcal{U}(\Lambda_{l_2,l_3}(t)) \equiv \Delta^{it/2\pi}_{\mathcal{R}_{l_\mathcal{L}}} \quad t \in \mathbb{R}.$$

Now define the observable algebra associated with arbitrary wedges by

$$\mathcal{R}(W[l_i, l_j]) \equiv ad\,\mathcal{U}(\Lambda_{l_i,l_j})(\mathcal{R}_{i_\mathcal{N}}) \ (\in \{\mathcal{R}_i\}_{i\in I} \text{ by the MSC and the CGMA}).$$

For translated wedges, define for $\alpha \in \mathbb{R}^{1,2}$

$$\mathcal{R}(W[l_i, l_j, \alpha]) \equiv ad\,\mathcal{U}(\alpha)\mathcal{U}(\Lambda_{l_i,l_j})(\mathcal{R}_{i_\mathcal{N}}) \ (\in \{\mathcal{R}_i\}_{i\in I} \text{ by the MSC and CGMA}).$$

In this way, for any wedge region W in $\mathbb{R}^{1,2}$ there is a unique von Neumann algebra $\mathcal{R}_{i_W}$ in $\{\mathcal{R}_i\}_{i\in I}$. Taking $F : \mathcal{W} \to I$ to be the map $F(W) = i_w$, for $W \in \mathcal{W}$ and $i_W \in I$ as obtained above and the result follows.

Conclusion 2 follows from Theorem 5.6 and Theorem 5.5. Conclusions 3 and 4 also follow from Theorem 5.6. The last conclusion follows from Theorem 4.3. ∎

We conclude this chapter with a few remarks. Given a net $\{\mathcal{A}_i\}_{i\in I}$ and a state ω satisfying purely algebraic conditions, one derives three-dimensional Minkowski space, and an identification between elements of I and the wedges in three-dimensional Minkowski space in such a way that the adJ_i act like a reflection through the spacelike edge of the wedge. Therefore, solely with assumptions on the algebras of observables $\{\mathcal{A}_i\}_{i\in I}$ and the preparation ω, we are able to derive the physical spacetime and its symmetries. We can also derive an interpretation of suitable elements of $\{\mathcal{A}_i\}_{i\in I}$ as local algebras associated with wedge regions, as well as derive a prescription of how the spacetime symmetries act upon the observables. In addition, we can get a time orientation of the spacetime from the MSC.

A similar process can be done for four-dimensional Minkowski space using the work of Wiesbrock and Kähler [18]. However, we refrain from giving the details here.

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BIOGRAPHICAL SKETCH

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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